10/3/2007

Theorem

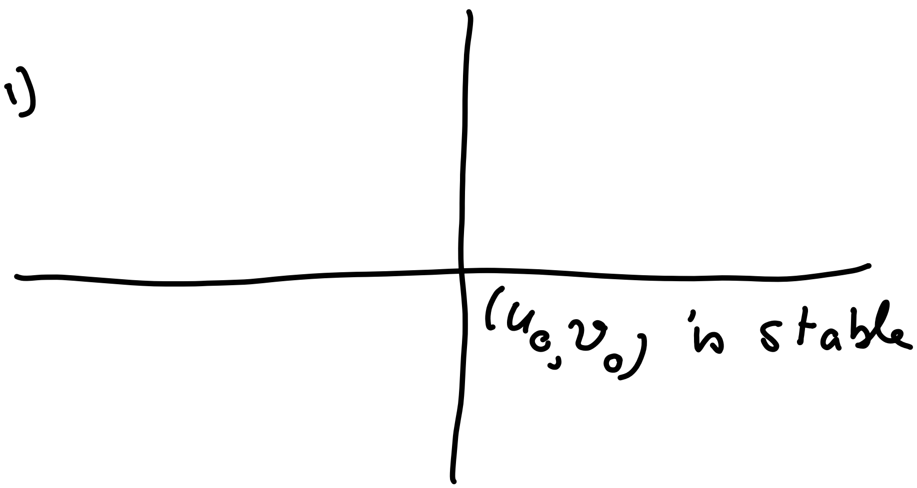
(i) $P_A = \max_u \text{ROWMIN}(u) \leq P_B = \min_v \text{COLMAX}(v)$

(ii) $\exists$ a stable solution iff $P_A = P_B$.

Proof

(i) ✓

(ii)

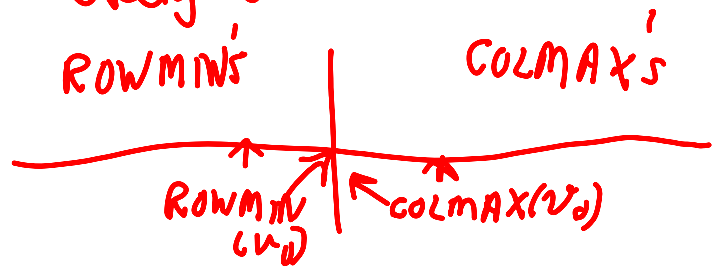


$\text{PAY}(u_0, v_0) = \text{ROWMIN}(u_0)$

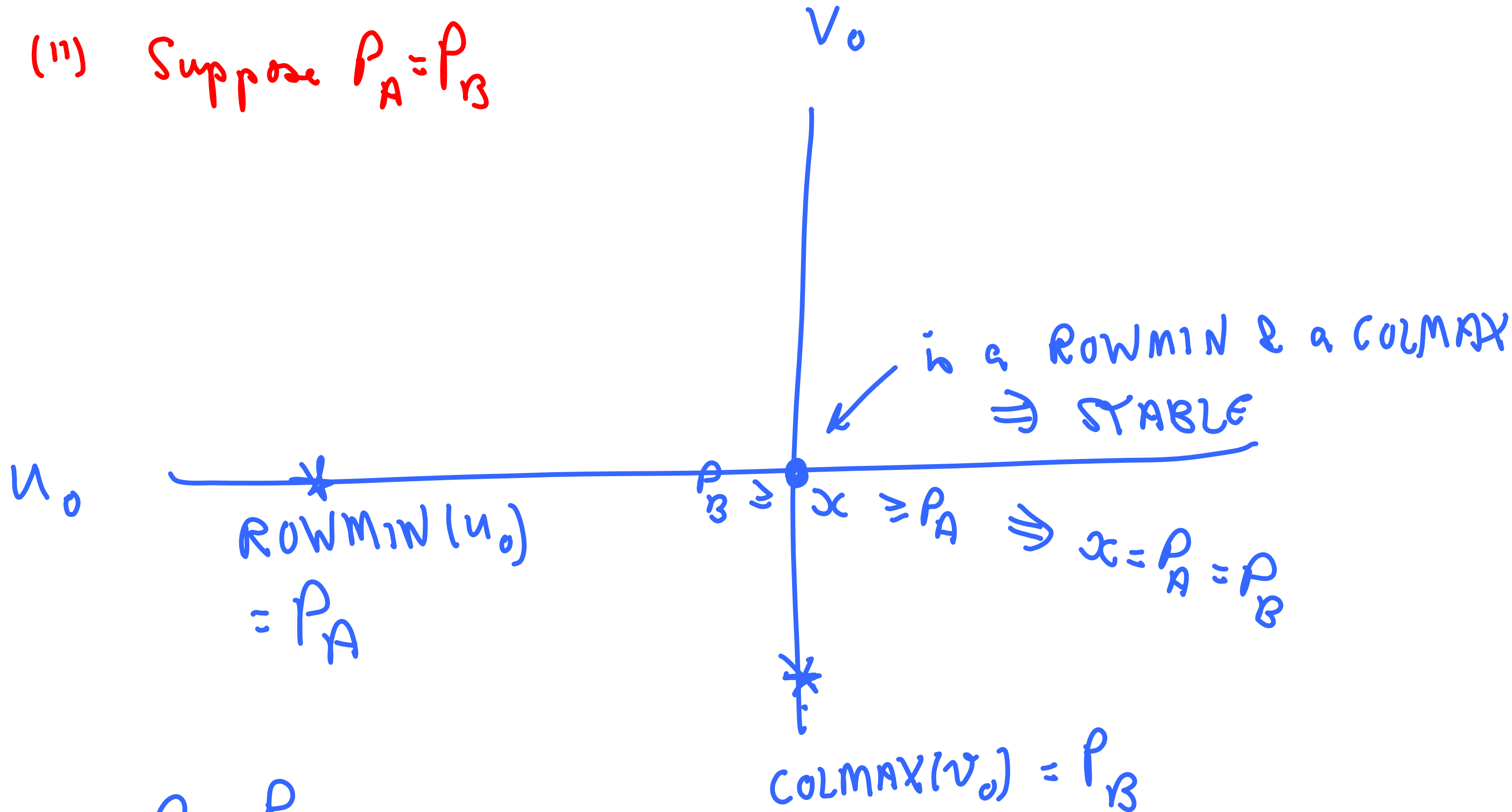
$= \text{COLMAX}(v_0)$

every ROWMIN $\leq$

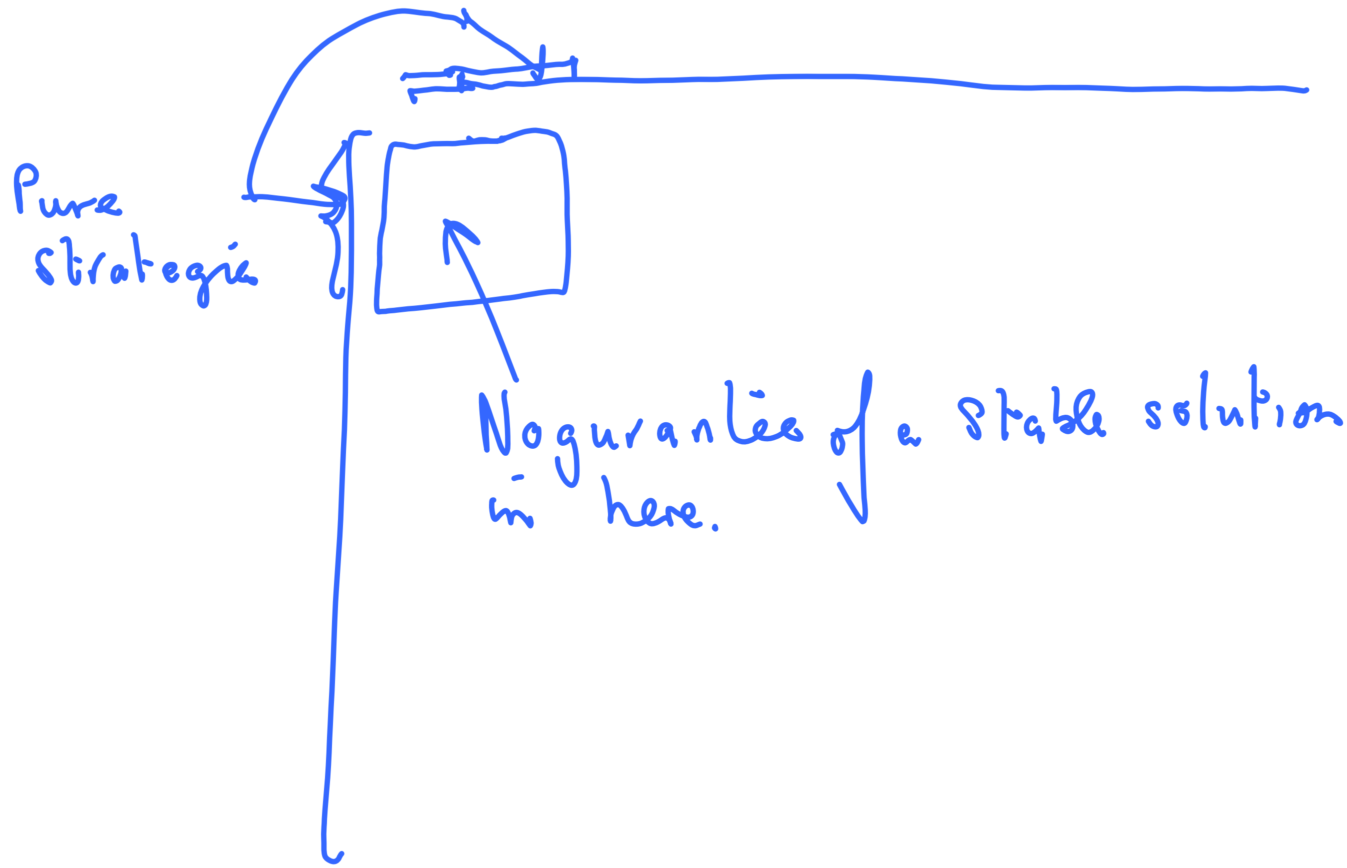
every COLMAX



(ii) Suppose $P_A = P_B$



$P_A = P_B$



Mixed Strategies:

A: $\underline{p} = (p_1, p_2, \dots, p_m)$ – play i with probability p_i.

B: $\underline{q} = (q_1, q_2, \dots q_n)$ – play j with probability q_j

$S_A = \{ \underset{\sim}{p} \}$ $S_B = \{ \underset{\sim}{q} \}$

$$\mathrm{PAY}(\underset{\sim}{p}, \underset{\sim}{q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, p_i\, q_j$$

= expected payoff.

Theorem

Now $P_A = P_B$

$$P_A = \max_{\underline{p}} \min_{\underline{q}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j$$

$$= \max_{\underline{p}} \min_{\substack{q_1+\cdots+q_n=1 \\ q_1,\ldots,q_n \geq 0}} \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{m} a_{ij} p_i \right)}_{c_j = c_j(\underline{p})} q_j$$

$$\min_{\substack{q_1+\cdots+q_n=1 \\ q_1,\ldots,q_n \geq 0}} \sum_{j=1}^{n} c_j q_j = \min \{ c_1, c_2, \ldots, c_n \}$$

e.g. $\min\limits_{\substack{q_1+q_2+q_3=1 \\ q_1,q_2,q_3 \geq 0}} \quad 3q_1 + 2q_2 + 4q_3 = 2(q_1+q_2+q_3) + q_1 + 2q_3$

$$= 2 + q_1 + 2q_3$$

$$P_A = \max_{\underline{p}} \min_{j=1,2,\cdots,n} \sum_{i=1}^{m} a_{ij} p_i$$

LP formulation:

Maximise ξ

$$\xi \leq \sum_{i=1}^{m} a_{ij} p_i \quad , \quad j=1,2,\cdots,n$$

$$p_1 + \cdots + p_m = 1$$

$$p_1, p_2, \cdots p_m \geq 0$$

$$P_B = \min_{\underline{q}} \max_{\underline{p}} \sum \sum a_{ij} p_i q_j$$

LP: minimise η

$$\eta \geq \sum_{j=1}^{n} a_{ij} q_j \qquad i = 1, 2, \dots, m$$

$$q_1 + \dots + q_n = 1$$

$$q_1, \dots q_n \geq 0$$

Maximise ξ

$$\xi \leq \sum_{i=1}^{m} a_{ij} p_i \quad , \quad j=1,2,\cdots,n$$

$p_1 + \cdots + p_m = 1$

$p_1, p_2, \cdots p_m \geq 0$

minimise η

$$\eta \geq \sum_{j=1}^{n} a_{ij} q_j \quad i=1,2,\cdots,m$$

$q_1 + \cdots + q_n = 1$

$q_1 \cdots q_n \geq 0$

Maximise

$c_1 x_1 + c_2 x_2 + c_3 x_3$

Dual

$y_1 \quad a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \leq b_1$

$y_2 \quad a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \geq b_2$

$y_3 \quad a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$

$x_1 \geq 0 \quad x_2 \leq 0$

$\Rightarrow$

Minimise

$b_1 y_1 + b_2 y_2 + b_3 y_3$

$a_{11} y_1 + a_{21} y_2 + a_{31} y_3 \geq c_1$

$a_{12} y_1 + a_{22} y_2 + a_{32} y_3 \leq c_2$

$a_{13} y_1 + a_{23} y_2 + a_{33} y_3 = c_3$

$y_1 \geq 0 \quad y_2 \leq 0$