Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 3: Due Monday October 23.

Q1 Solve the following 2-person zero-sum games:

$$\begin{bmatrix} 6 & 2 & 4 \\ 5 & 2 & 5 \\ 4 & 1 & -3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 1 & 0 & -1 \\ 4 & 3 & 2 & 1 & -1 \\ 1 & 1 & 0 & -1 & 1 \\ 2 & 1 & 1 & -2 & -2 \\ 4 & 1 & 0 & -2 & -3 \end{bmatrix}$$

Solution (2,2) is a saddle point for the first game. Thus the solution is for player A to use 1 and player B to use 2. The value of the game is 2.
For the second game we have the following sequence of row/column removals because of domination:

$$\text{Remove column strategy 1.} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 3 & 2 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & -2 & -2 \\ 1 & 0 & -2 & -3 \end{bmatrix}$$

$$\text{Remove column strategy 2.} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & -2 & -2 \\ 0 & -2 & -3 \end{bmatrix}$$

$$\text{Remove column strategy 3.} \begin{bmatrix} 0 & -1 \\ 1 & -1 \\ -1 & 1 \\ -2 & -2 \\ -2 & -3 \end{bmatrix}.$$

Remove row strategy 1. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -2 & -2 \\ -2 & -3 \end{bmatrix}$

Remove row strategy 4. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -2 & -3 \end{bmatrix}$

Remove row strategy 5. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

The optimal strategies for this game are for player A to play rows 2 and 3 with probability 1/2 each. Similarly, player B plays columns 4 and 5 with probabilty 1/2 each.
Q2 Find a symmetric equilibrium for the first price sealed bid auction in the case where $N = 3$ and $F(x) = 1 - e^{-\lambda x}$.

Solution In this case,

$$\begin{aligned} \beta(x) &= x - \int_{y=0}^{x} \left(\frac{F(y)}{F(x)}\right)^2 dy \\ &= x - \int_{y=0}^{x} \left(\frac{1 - e^{-\lambda y}}{1 - e^{-\lambda x}}\right)^2 dy \\ &= x + \frac{1 - 4e^{-\lambda x} + e^{2\lambda x}(3 - 2\lambda x)}{2\lambda(1 - e^{-\lambda x})^2}. \end{aligned}$$

Q3 A firm manufactures and sells 3 different products, production time. Data on costs and demands are given in the table below. The maximum average inventory that can be held is \$4000. Find the optimal re-order policy under these circumstances if the inventory charge is 10% of the items value per period.

Product	Demand/Year	Value/Item	Set up cost
1	200	10	50
2	1500	15	40
3	300	20	60

Solution If the firm orders quantities Q_1, Q_2, Q_3 at a time, then the total cost is

$$T(Q_1, Q_2, Q_3) = \frac{10000}{Q_1} + \frac{Q_1}{2} + \frac{60000}{Q_2} + \frac{3Q_2}{4} + \frac{18000}{Q_3} + Q_3.$$

The unconstrained (Wilson Lot-Size) optimum is

$$Q_1 = \sqrt{20000},\ Q_2 = \sqrt{80000},\ Q_3 = \sqrt{18000}.$$

This gives an average inventory value of

$$\frac{10Q_1 + 15Q_2 + 20Q_3}{2} = \sqrt{20000} \times 5 + \sqrt{80000} \times 15/2 + \sqrt{18000} \times 10 > 4000.$$

So we must minimise $T(Q_1, Q_2, Q_3)$ subject to $10Q_1 + 15Q_2 + 20Q_3 = 8000$. Let λ be the Lagrange multiplier. Then we have

$$\begin{aligned}
-\frac{10000}{Q_1^2} + \frac{1}{2} + 10\lambda &= 0. \\
-\frac{60000}{Q_2^2} + \frac{3}{4} + 15\lambda &= 0. \\
-\frac{18000}{Q_3^2} + 1 + 20\lambda &= 0.
\end{aligned}$$

We must then choose λ so that $10Q_1 + 15Q_2 + 20Q_3 = 8000$ or

$$10\sqrt{\frac{10000}{1/2 + 10\lambda}} + 15\sqrt{\frac{60000}{3/4 + 15\lambda}} + 20\sqrt{\frac{18000}{1 + 20\lambda}} = 8000.$$

Solving give $\lambda \approx .00434$ and

$$Q_1 \approx 135.656,\ Q_2 \approx 271.313,\ Q_3 \approx 128.695.$$