Department of Mathematical Sciences
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OPERATIONS RESEARCH II 21-393

Homework 2.

Q1 Solve the following 2-person zero-sum games:

$$\begin{bmatrix} 6 & 2 & 4 \\ 5 & 2 & 5 \\ 4 & 1 & -3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 1 & 0 & -1 \\ 4 & 3 & 2 & 1 & -1 \\ 1 & 1 & 0 & -1 & 1 \\ 2 & 1 & 1 & -2 & -2 \\ 4 & 1 & 0 & -2 & -3 \end{bmatrix}$$

Solution (2,2) is a saddle point for the first game. Thus the solution is for player A to use 1 and player B to use 2. The value of the game is 2.
For the second game we have the following sequence of row/column removals because of domination:

$$\text{Remove column strategy 1.} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 3 & 2 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & -2 & -2 \\ 1 & 0 & -2 & -3 \end{bmatrix}$$

$$\text{Remove column strategy 2.} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & -2 & -2 \\ 0 & -2 & -3 \end{bmatrix}$$

$$\text{Remove column strategy 3.} \begin{bmatrix} 0 & -1 \\ 1 & -1 \\ -1 & 1 \\ -2 & -2 \\ -2 & -3 \end{bmatrix}.$$

$$\text{Remove row strategy 1.} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -2 & -2 \\ -2 & -3 \end{bmatrix}$$

Remove row strategy 4. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -2 & -3 \end{bmatrix}$

Remove row strategy 5. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

The optimal strategies for this game are for player A to play rows 2 and 3 with probability 1/2 each. Similarly, player B plays columns 4 and 5 with probabilty 1/2 each.

Q2 Suppose the $n \times n$ matrix A is such that all row and column sums are equal to the same value C. What is the solution to this game?

Solution We can assume that $C > 0$. We know that

$$P_A^{-1} = \min x_1 + x_2 + \cdots + x_n : \ A^T\mathbf{x} \geq \mathbf{1},\ \mathbf{x} \geq 0.$$

$$P_B^{-1} = \min y_1 + y_2 + \cdots + y_n : \ A\mathbf{y} \geq \mathbf{1},\ \mathbf{y} \geq 0.$$

Next observe that $x_i = y_j = C^{-1}$ for all i, j gives feasible solutions to these dual problems with the same value nC^{-1}. It follows that $P_A = P_B = C/n$ and that the optimal strategy for each palyer is to play each strategy with probabilty $1/n$.

Q3 Formulate the following problems as integer programs:

(a) The government has asked for and received bids on m construction projects from each of n firms. No firm will be awarded more than one contract and for political reasons no more than p large contracts are to go to foreign firms. Projects $1, 2, \ldots, \ell$ are large and firms $1, 2, \ldots, f$ are foreign. If $b_{i,j}$ is the bid by firm i for project j, which bids should be accepted to minimise the total cost?

Solution

$$
\begin{aligned}
min \sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij}x_{ij} & \\
s.t. & \\
\sum_{i=1}^{n} x_{ij} &= 1 \quad \forall j \\
\sum_{j=1}^{m} x_{ij} &\leq 1 \quad \forall i \\
\sum_{i=1}^{f}\sum_{j=1}^{l} x_{ij} &\leq p \\
x_{ij} &\in \{0,1\}
\end{aligned}
$$

The first constraint ensures that each project is assigned to some firm. The second that no firm gets more than one project and the third that the foreign firms get less than p.

(b) For the purpose of fire safety, a town is divide into n areas. The council has decided to build p fire stations. m possible sites have been found. Let $t_{i,j}$ denote the time taken to drive from area i to area j. The cost of locating a fire station at location i is f_i. Each area must be within driving time τ of a fire station. Where should the fire stations be located to minimise cost? [**Hint:** Its a set-covering problem.]

Solution Define the set

$$S_j = \{i | t_{ij} \leq \tau\}$$

S_j consists of all those locations that are within distance τ of area j.

$$
\begin{aligned}
min \ \sum_{i=1}^{m} f_i x_i & \\
s.t. & \\
\sum_{i \in S_j} x_i & \geq 1 \quad \forall j \in \{1, \ldots, n\} \\
\sum_{i=1}^{m} x_i & = p \\
x_i & \in \{0, 1\}
\end{aligned}
$$

The first constraint ensures that every area is within distance τ of some fire station. The second ensures that there are no more than p fire stations constructed.

(c) An assembly line consists of a sequence of locations called work stations. The manufacture of a certain object requires m separate jobs to be undertaken with job i requiring t_i minutes. The jobs are to be allocated to work stations so that each station completes a set of jobs and then passes the object onto the next station on the line and waits to receive the next object from the previous station on the line. The combined time of all jobs assigned to any station must not exceed T the cycle time. Also there are a number of precedence relations between jobs indicated by the digraph $D = (V, A)$ where $(i, j) \in A$ if job i must precede job j. The problem is to open as few work stations as possible consistent with the cycle time.

Solution

$$
\begin{aligned}
min \sum_{i=1}^{m} y_i & & \\
s.t. & & \\
x_{ij} & \leq & y_i \quad \forall i,j \in \{1,\ldots,m\} \\
\sum_{j=1}^{m} x_{ij} t_j & \leq T \quad \forall i \in \{1,2,\ldots,m\} & \\
\sum_{i=1}^{m} x_{ij} & = & 1 \quad \forall j \in \{1,2,\ldots,m\} \\
\sum_{i=1}^{m} i x_{ij_1} & \leq & \sum_{i=1}^{m} i x_{ij_2} \quad \forall (j_1,j_2) \in A \\
x_{ij} & \in & \{0,1\} \forall i,j \in \{1,\ldots,m\} \\
y_i & \in & \{0,1\} \forall i \in \{1,\ldots,m\}
\end{aligned}
$$

x_{ij} is 1 when job j is done at station i and 0 otherwise.
y_i is 1 if at least one job is done at station i.
The first constraint ensures that if any job is done at station i, the variable y_i is 1.
The second constraint ensures that each station satisfies the cycle time T.
The third constraint ensures that each job is scheduled on some machine.
The last constraint ensures the j_1 is done before job j_2 if there is a precedence constraint between them.
Q4 Solve the following problem by a cutting plane algorithm:

$$
\begin{array}{lllllll}
\text{minimise} & 4x_1 & + & 5x_2 & +3x_3 & & \\
\text{subject to} & & & & & & \\
 & 2x_1 & + & x_2 & -x_3 & \geq & 2 \\
 & x_1 & + & 4x_2 & +x_3 & \geq & 13
\end{array}
$$

$$x_1, x_2, x_3 \geq 0 \text{ and integer.}$$

Solution
Initial tableau

x_1	x_2	x_3	x_4	x_5	R.H.S	
-4	-5	-3	0	0	0	z
-2	-1	1	1	0	-2	x_4
-1	-4	-1	0	1	-13	$x_5 \leftarrow$
	$\uparrow$					

x_1	x_2	x_3	x_4	x_5	R.H.S	
$\frac{-11}{4}$	0	$\frac{-7}{4}$	0	$\frac{-5}{4}$	$\frac{65}{4}$	z
$\frac{-7}{4}$	0	$\frac{5}{4}$	1	$\frac{-1}{4}$	$\frac{5}{4}$	x_4
$\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{-1}{4}$	$\frac{13}{4}$	x_2

Primal feasible, but the solution is not integral.
We add a cut which eliminates the current optimal solution.

$$\frac{1}{4}x_1 + \frac{1}{4}x_3 + \frac{3}{4}x_5 - y_1 = \frac{1}{4}$$

x_1	x_2	x_3	x_4	x_5	y_1	R.H.S	
$\frac{-11}{4}$	0	$\frac{-7}{4}$	0	$\frac{-5}{4}$	0	$\frac{65}{4}$	z
$\frac{-7}{4}$	0	$\frac{5}{4}$	1	$\frac{-1}{4}$	0	$\frac{5}{4}$	x_4
$\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{-1}{4}$	0	$\frac{13}{4}$	x_2
$\frac{-11}{4}$	0	$\frac{-1}{4}$	0	$\frac{-3}{4}$	+1	$\frac{-1}{4}$	$y_1 \leftarrow$
				$\uparrow$			

We do a dual simplex pivot to obtain

x_1	x_2	x_3	x_4	x_5	y_1	R.H.S	
$\frac{-7}{3}$	0	$\frac{-4}{3}$	0	$\frac{-5}{3}$	0	$\frac{50}{3}$	z
$\frac{-5}{3}$	0	$\frac{4}{3}$	1	0	$\frac{-1}{3}$	$\frac{4}{3}$	x_4
$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	$\frac{-1}{3}$	$\frac{10}{3}$	x_2
$\frac{1}{3}$	0	$\frac{1}{3}$	0	1	$\frac{-4}{3}$	$\frac{1}{3}$	x_5

The solution is primal feasible and so optimal but still not integer.
We add a cut which eliminates the current optimal solution.

$$\frac{-1}{3}x_1 - \frac{1}{3}x_3 + y_2 = \frac{1}{3}$$

We obtain tableau

x_1	x_2	x_3	x_4	x_5	y_2	R.H.S	
$\frac{-7}{3}$	0	$\frac{-4}{3}$	0	0	0	$\frac{50}{3}$	z
$\frac{-5}{3}$	0	$\frac{4}{3}$	1	0	0	$\frac{4}{3}$	x_4
$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	0	$\frac{10}{3}$	x_2
$\frac{1}{3}$	0	$\frac{1}{3}$	0	1	0	$\frac{1}{3}$	x_5
$\frac{-1}{3}$	0	$\frac{-1}{3}$	0	0	1	$\frac{1}{3}$	$y_2 \leftarrow$
		$\uparrow$					

We do a dual simplex pivot to obtain

x_1	x_2	x_3	x_4	x_5	y_2	R.H.S	
-1	0	0	0	0	-4	18	z
-3	0	0	1	0	4	0	x_4
0	1	0	0	0	1	3	x_2
0	0	0	0	1	1	0	x_5
1	0	1	0	0	-3	1	x_3

Which is optimal integral.

Q5 Solve the following problem by a branch and bound algorithm:

$$\begin{array}{lllllll}
\text{Maximise} & 4x_1 & -2x_2 & +7x_3 & -x_4 & \\
\text{subject to} & & & & & \\
& x_1 & & +5x_3 & & \leq 10 \\
& x_1 & +x_2 & -x_3 & & \leq 1 \\
& 6x_1 & -5x_2 & & & \leq 0 \\
& -x_1 & & +2x_3 & -2x_4 & \leq 3
\end{array}$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

$$x_1, x_2, x_3 \text{ integer.}$$

Solution

1. LP relaxation:

$$(x_1, x_2, x_3, x_4) = \left(\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 0\right) \qquad Value = 14\frac{1}{4}.$$

Sub-problem 1: add constraint $x_1 \leq 1$.

$$(x_1, x_2, x_3, x_4) = \left(1, \frac{6}{5}, \frac{9}{5}, 0\right) \qquad Value = 14\frac{1}{5}.$$

Sub-problem 2: add constraint $x_1 \geq 2$.
No solutions.

Subproblem 1.1: add constraint $x_2 \leq 1$.

$$(x_1, x_2, x_3, x_4) = \left(\frac{5}{6}, 1, \frac{11}{6}, 0\right) \qquad Value = 14\frac{1}{6}.$$

Subproblem 1.2: add constraint $x_2 \geq 2$.

$$(x_1, x_2, x_3, x_4) = \left(\frac{5}{6}, 2, \frac{11}{6}, 0\right) \qquad Value = 12\frac{1}{6}.$$

Sub-problem 1.1.1: add constraint $x_1 \leq 0$.

$$(x_1, x_2, x_3, x_4) = \left(0, 0, 2, \frac{1}{2}\right) \qquad Value = 13\frac{1}{2}.$$

This solution is feasible.
Subproblem 1.1.2: add constraint $x_1 \geq 1$. No solutions.
Sub-problem 1.2 is *fathomed* i.e. there is no solution to this problem which is better than our current *incumbent*.
Optimal solution: $(x_1, x_2, x_3, x_4) = \left(0, 0, 2, \frac{1}{2}\right) \qquad Value = 13\frac{1}{2}$.