Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Answers to homework 1.

Q1 Solve the following knapsack problem:

$$\begin{array}{lccccccc} \text{maximise} & 2x_1 & + & 6x_2 & + & 8x_3 & & \\ \text{subject to} & & & & & & & \\ & 2x_1 & + & 4x_2 & + & 5x_3 & \leq & 15 \end{array}$$

$$x_1, x_2, x_3 \geq 0 \text{ and integer.}$$

Solution:

w	f_1	δ_1	f_2	δ_2	f_3	δ_3
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	2	1	2	0	2	0
3	2	1	2	0	2	0
4	4	1	6	1	6	0
5	4	1	6	1	8	1
6	6	1	8	1	8	0/1
7	6	1	8	1	10	1
8	8	1	12	1	12	0
9	8	1	12	1	14	1
10	10	1	14	1	16	1
11	10	1	14	1	16	1
12	12	1	18	1	18	1
13	12	1	18	1	20	1
14	14	1	20	1	22	1
15	14	1	20	1	24	1

Solution: $x_1 = 0, x_2 = 0, x_3 = 3$. Maximum = 24.
Start with $x_1 = x_2 = x_3 = 0$. $\delta_3(21) = 1$ and so we add one to x_3. We have used up 5 units of the knapsack. There are 10 units left. $\delta_3(10) = 1$ and so we add one to x_3. We use up another 5 units and so we are left with 5. $\delta_3(5) = 1$. We add one more to x_3. There are now 0 units in the knapsack. $\delta_3(0) = 0$ and so we move to column 2. $\delta_2(0) = 0$ and so we move to column 1. $\delta(0) = 0$ and we are done.

Q2:A county chairwoman of a certain political party is making plans for an upcoming presidential election. She has received the services of 10 volunteer workers for precinct work and wants to assign them to five precincts in such a way as to maximize their effectiveness. She feels that it would be inefficient to assign a worker to more than one precinct, but she is willing to assign no workers to any one of the precincts if they can accomplish more in other precincts.
The following table gives the estimated increase in the number of votes for the party's candidate in each precinct if it were allocated the various number of workers.

Number of Workers	Precinct 1	2	3	4	5
0	0	0	0	0	0
1	4	7	5	6	4
2	10	11	10	11	12
3	15	16	15	14	15
4	18	18	18	16	17
5	22	20	21	17	20
6	24	21	22	18	22
7	26	25	24	23	22
8	28	27	27	25	24
9	32	25	30	28	26
10	33	28	34	30	29

Use dynamic programming to find all solutions to the problem of maximising votes.
Solution: Let $f_r(w)$ be the maximum increase in votes in precincts $1, 2, \ldots, r$ assuming that w workers are assigned to them.

$$f_r(w) = \max_{0 \leq i \leq w} \{a_{i,r} + f_{r-1}(w-i)\}, \qquad r \geq 1,$$

where $a_{i,r}$ is the increased number of votes gained from i workers in precinct r.

$$f_0(w) = 0.$$

w	f_1	x_1	f_2	x_2	f_3	x_3	f_4	x_4	f_5	x_5
0	0	0	0	0	0	0	0	0		
1	4	1	7	1	7	0	7	0		
2	10	2	11	1,2	13	1	13	1		
3	15	3	17	1	17	0,2	18	1,2		
4	18	4	22	1	22	0,1,3	28	1,2		
5	22	5	26	2	27	1,2	28	1,2		
6	24	6	31	3	32	2,3	33	1,2		
7	26	7	34	3	37	3	38	1,2		
8	28	8	38	3	41	2,3	43	1,2		
9	32	9	40	3,4	46	3	48	2		
10	33	10	42	3,4,5	49	3,4	52	1,2	55	2

Solutions: Maximum = 55.

$$\begin{array}{lllll} x_1 = 3 & x_2 = 1 & x_3 = 3 & x_4 = 1 & x_5 = 2 \\ x_1 = 3 & x_2 = 1 & x_3 = 2 & x_4 = 2 & x_5 = 2 \\ x_1 = 2 & x_2 = 1 & x_3 = 3 & x_4 = 2 & x_5 = 2 \end{array}$$

Q3: The people of a certain area live at the side of a long straight road of length L. The population is clustered into several villages at points $a_1, a_2, \ldots,$ a_k along the road. There is a proposal to build ℓ fire stations on the road. The problem is build them so that the maximum distance of a village to its nearest fire station is minimised. Formulate the problem of finding the optimum placement of fire stations as a dynamic program. (Assume that fire stations are to be placed at integer points only on the line.)

Solution: Let

$$c(i, j : a) = \max_{k : i \leq a_k \leq j} |a_k - a|$$

be the maximum distance of a village in $[i, j]$ to a firestation serving $[i, j]$ if it is placed at a. Then let

$$c(i, j) = \min\{c(i, j : a) : \ i \leq a \leq j\}.$$

Let $f(x, i)$ be the maximum distance from a village in $[0, x]$ to its nearest fire station in $[0, x]$ if i fire stations are optimally placed to service the villages in $[0, x]$. Then $f(i, i) = 0$ for $i = 0, 1, 2, \ldots, \ell$ and

$$f(x, i) = \min_{i \leq y \leq x} \{\max\{f(y, i-1), c(y, x)\}\}.$$

Here y is the tentative place to put the ith firestation.

Q4: Let Σ be a k letter alphabet and let $v = v_1 v_2 \cdots v_m, w = w_1 w_2 \cdots w_n$ be two sequences over Σ. The distance $d(v, w)$ between v and w is defined as follows: A padding π of v is the insertion of a number of $\star$ symbols into the sequence. For example $A \star CGC \star T \star G$ is a padding of $ACGCTG$. Now let $x = x_1 x_2 \cdots x_p, y = y_1 y_2 \cdots y_p$ be two sequence of the same length over $A \cup \{\star\}$. Let

$$\delta(x, y) = \sum_{i=1}^{p} \theta(x_i, y_i)$$

where

$$\theta(x_i, y_i) = \begin{cases} 0 & x_i = y_i \neq \star \\ 1 & x_i = y_i = \star \\ a & x_i \neq y_i, \{x_i, y_i\} \subseteq \Sigma \\ b & x_i \in \Sigma \text{ and } y_i = \star \text{ or } x_i = \star \text{ and } y_i \in \Sigma \end{cases}.$$

Then

$$d(v, w) = \min\{\delta(x, y) : \ x, y \text{ are paddings of } v, w \text{ respectively}\}.$$

Show how to use dynamic programming to compute $d(v, w)$.
(When the alphabet $\Sigma = \{A, C, G, T\}$, this is an important problem in Computational Biology).

Solution: Let

$$f(i, j) = d(v_1 v_2 \cdots v_i, w_1 w_2 \cdots w_j).$$

Then

$$f(i, j) = \min \begin{cases} f(i-1, j) + b \\ f(i, j-1) + b \\ f(i-1, j-1) + a \times 1_{v_i \neq w_j} \end{cases}$$

and

$$f(1, 0) = f(0, 1) = b \text{ and } f(1, 1) = a \times 1_{v_1 \neq w_1}$$

Q5: Consider an electronic system consisting of four components each of which must work for the system to function. The reliability of the system can be improved by installing several parallel units in one or more of the components. The following table gives the probability that the respective components will function if they consist of one, two or three parallel units.

No. of units	Component 1	Component 2	Component 3	Component 4
1	0.4	0.6	0.7	0.5
2	0.6	0.7	0.8	0.7
3	0.8	0.8	0.9	0.9

The probability that the system will function is the product of the probabilities that the respective components will function. The cost (in hundred's of dollars) of installing one, two or three parallel units in the respective compoonents is given by the following table.

No. of units	Component 1	Component 2	Component 3	Component 4
1	1	2	1	1
2	2	4	3	3
3	3	5	4	4

Because of budget limitations, a maximum of \$1000 can be spent. use Dynamic Programming to determine how many parallel units should be installed in each of the four components in order to maximise the probability that the system will function.

Solution: Let $p_i(x)$ be the probability that component i functions if there are x parallel units placed in it and let $c_r(x)$ be the cost of putting x parallel units into component i. Let $\phi_r(w)$ be the maximum reliability of a system of components $r, r+1, \ldots, 4$ if one has enough money to put in w parallel components altogether. We wish to compute $\phi_1(10)$. In general

$$
\begin{aligned}
\phi_4(w) &= p_4(w) \\
\phi_r &= \max_x (p_r(x)\phi_{r+1}(w - c_r(x)))
\end{aligned}
$$

w	ϕ_1	x_1	ϕ_2	x_2	ϕ_3	x_3	ϕ_4	x_4
0			0		0		0	
1			0		0		0	
2			0		.35	1	.5	1
3			0		.35	1	.7	2
4			0		.49	1	.9	3
5			.210	1	.63	1	.9	3
6			.294	1	.63	1	.9	3
7			.378	1	.72	2	.9	3
8			.378	1	.81	3	.9	3
9			.441	2	.81	3	.9	3
10	.3024	3	.504	3	.81	3	.9	3

Solution: $x_1 = 3, x_2 = 1, x_3 = 1, x_4 = 3$. Maximum $= .3024$.

Q6 Suppose that the price of a *beanie baby* changes from day to day; on a given day it is equally to \$$x$ where $x \in \{5, 6, 7, 8, 9, 10\}$. Assume also that the price on any day is independent of the price on any other day.

1. If today's price is \$8 and you must purchase a beanie baby for your niece's birthday party tomorrow evening, should you wait until tomorrow?

2. What is the best strategy with 5 days to go?

3. Suppose that the Beanie outlet advertises an offer: Under this offer, you can buy a beanie baby at a fixed price of \$6 any time during the next five days. What is the maximum you would be the maximum you would be prepared to pay for this option.

It is assumed that you wish to minimise the expected price that you pay for the beanie baby.

Solution: Let f_n denote the minimum expected cost of buying a beanie baby if there are n days to go. Then $f_1 = (5 + 10)/2 = 15/2$ and

$$f_n = \frac{1}{6} \sum_{p=5}^{10} \min\{p, f_{n-1}\}. \tag{1}$$

1. $f_1 < 8$ and so you should not buy today.

2.

$$\begin{aligned}
f_1 &= \frac{15}{2}.\\
f_2 &= \frac{1}{6}\left(5+6+7+\frac{15}{2}+\frac{15}{2}+\frac{15}{2}\right)=\frac{27}{4}\\
f_3 &= \frac{1}{6}\left(5+6+\frac{27}{4}+\frac{27}{4}+\frac{27}{4}+\frac{27}{4}\right)=\frac{19}{3}\\
f_4 &= \frac{1}{6}\left(5+6+\frac{19}{3}+\frac{19}{3}+\frac{19}{3}+\frac{19}{3}\right)=\frac{109}{18}\\
f_5 &= \frac{1}{6}\left(5+6+\frac{109}{18}+\frac{109}{18}+\frac{109}{18}+\frac{109}{18}\right)=\frac{317}{54}
\end{aligned}$$

Strategy: Buy at 5 or 6 for first 3 days and then at 5 or 6 or 7 for next day. Must buy on last day.

3. replace (1) by

$$f_n = \frac{1}{6}\sum_{p=5}^{10}\min\{p, 6, f_{n-1}\}.$$

Then

$$\begin{aligned}
f_1 &= 6.\\
f_2 &= \frac{1}{6}(5+6+6+6+6+6)=\frac{35}{6}\\
f_3 &= \frac{1}{6}\left(5+\frac{35}{6}+\frac{35}{6}+\frac{35}{6}+\frac{35}{6}+\frac{35}{6}\right)=\frac{205}{36}\\
f_4 &= \frac{1}{6}\left(5+\frac{205}{36}+\frac{205}{36}+\frac{205}{36}+\frac{205}{36}+\frac{205}{36}\right)=\frac{1205}{216}\\
f_5 &= \frac{1}{6}\left(5+\frac{1205}{216}+\frac{1205}{216}+\frac{1205}{216}+\frac{1205}{216}+\frac{1205}{216}\right)=\frac{7105}{1296}
\end{aligned}$$

So the offer is worth $\frac{317}{54}-\frac{7105}{1296}$.

Q7 A mathematics student has been asked to compute the product $A_1 \times A_2 \times \cdots \times A_n$ where each A_i is an $m_i \times m_{i+1}$ matrix. It will cost the student $\$pqr$ to multiply a $p \times q$ matrix by a $q \times r$ matrix. Describe a dynamic programming algorithm for finding the cheapest way of multiplying the n matrices together.

To see that there is a problem: let $\underline{m} = (10, 5, 8, 12)$. There are two ways to compute $A_1A_2A_3$: compute A_1A_2 first and then multiply this by A_3 – total cost 400+960=1360. The other way is to compute A_2A_3 first and then multiply this by A_1 – total cost 480+600=1080.

Solution: Let $M_{i,j}$ denote the minimum number of multiplications needed to compute $A_i \times A_{i+1} \times \cdots \times A_j$ for $1 \leq i < j \leq n$. Then, by considering the last matrix multiplication needed,

$$M_{i,j} = \min_{i \leq k < j} \{M_{i,k} + M_{k+1,j} + m_i m_{k+1} m_{j+1}\}.$$

We can use this recurrence to compute $M_{i,n}$, computing the terms $M_{i,j}$ in order of the sum $i + j$.

Q8 A system can be in 3 states 1,2,3 and the cost of moving from state i to state j in one period is $c(i,j)$, where the $c(i,j)$ are given in the matrix below. The one period discount factor α is 1/2.

The aim is to find a policy which simultaneously minimises the discounted cost of operating from any starting state. Start with the policy

$$\pi(1) = 1, \pi(2) = 1, \pi(3) = 2.$$

Find an optimal policy.

The matrix of costs is

$$\begin{bmatrix} 7 & 2 & 3 \\ 5 & 2 & 7 \\ 1 & 6 & 2 \end{bmatrix}$$

Solution: Start with an initial solution of $\pi(1) = 2, \pi(2) = 2, \pi(3) = 1$ i.e. chose the minimum cost in each row (a reasonable starting solution).

Then we solve

$$\begin{aligned} y_1 &= 2 + \frac{y_2}{2} \\ y_2 &= 2 + \frac{y_2}{2} \\ y_3 &= 1 + \frac{y_1}{2} \end{aligned}$$

Solving these equations gives $y_1 = 4, y_2 = 4, y_3 = 3$.

We now check for optimality:

Check y_1:

$$
\begin{aligned}
7 &+ \frac{y_1}{2} = 9 \\
2 &+ \frac{y_2}{2} = 4 \qquad \star \\
3 &+ \frac{y_3}{2} = \frac{13}{2}
\end{aligned}
$$

Check y_2:

$$
\begin{aligned}
5 &+ \frac{y_1}{2} = 7 \\
2 &+ \frac{y_2}{2} = 4 \qquad \star \\
7 &+ \frac{y_3}{2} = \frac{17}{2}
\end{aligned}
$$

Check y_3:

$$
\begin{aligned}
1 &+ \frac{y_1}{2} = 3 \quad \star \\
6 &+ \frac{y_2}{2} = 8 \\
2 &+ \frac{y_3}{2} = \frac{7}{2}
\end{aligned}
$$

So our initial guess is optimal.