Department of Mathematical Sciences
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OPERATIONS RESEARCH II 21-393

Answers to homework 2.

Q1 Solve the following 2-person zero-sum games:

$$\begin{bmatrix} 6 & 2 & 4 \\ 5 & 2 & 5 \\ 4 & 1 & -3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 1 & 0 & -1 \\ 4 & 3 & 2 & 1 & -1 \\ 1 & 1 & 0 & -1 & 1 \\ 2 & 1 & 1 & -2 & -2 \\ 4 & 1 & 0 & -2 & -3 \end{bmatrix}$$

Solution (2,2) is a saddle point for the first game. Thus the solution is for player A to use 1 and player B to use 2. The value of the game is 2.
For the second game we have the following sequence of row/column removals because of domination:

$$\text{Remove column strategy 1.} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 3 & 2 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & -2 & -2 \\ 1 & 0 & -2 & -3 \end{bmatrix}$$

$$\text{Remove column strategy 2.} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & -2 & -2 \\ 0 & -2 & -3 \end{bmatrix}$$

$$\text{Remove column strategy 3.} \begin{bmatrix} 0 & -1 \\ 1 & -1 \\ -1 & 1 \\ -2 & -2 \\ -2 & -3 \end{bmatrix}.$$

$$\text{Remove row strategy 1.} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -2 & -2 \\ -2 & -3 \end{bmatrix}$$

Remove row strategy 4. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -2 & -3 \end{bmatrix}$

Remove row strategy 5. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

The optimal strategies for this game are for player A to play rows 2 and 3 with probability 1/2 each. Similarly, player B plays columns 4 and 5 with probabilty 1/2 each.

Q2: Suppose the $n \times n$ matrix A is such that all row and column sums are equal to the same value C. What is the solution to this game?

Solution We can assume that $C > 0$. We know that

$$P_A^{-1} = \min x_1 + x_2 + \cdots + x_n : \ A^T\mathbf{x} \geq \mathbf{1}, \ \mathbf{x} \geq 0.$$

$$P_B^{-1} = \min y_1 + y_2 + \cdots + y_n : \ A\mathbf{y} \geq \mathbf{1}, \ \mathbf{y} \geq 0.$$

Next observe that $x_i = y_j = C^{-1}$ for all i, j gives feasible solutions to these dual problems with the same value nC^{-1}. It follows that $P_A = P_B = C/n$ and that the optimal strategy for each palyer is to play each strategy with probabilty $1/n$.

Q3 The correlation coefficient between assets A and B is .1 and the other data is given below:

ASSET	$\bar{r}$	σ
A	.1	.15
B	.18	.3

(a) Find the proportions α of A and $1-\alpha$ of B that define a portfolio having minimum standard deviation.

(b) What is the value of this minimum standard deviation.

(c) What is the expected returm for this portfolio.

Solution (a) Let $\sigma(\alpha)$ be the standard deviation of $\alpha A + (1-\alpha)B$. Then

$$\sigma(\alpha)^2 = .0225\alpha^2 + .009\alpha(1-\alpha) + .09(1-\alpha)^2.$$

This is minimisd when

$$.045\alpha + .009 - .018\alpha - .18(1-\alpha) = 0$$

or

$$\alpha = \frac{171}{207}.$$

(b)

$$\sigma(171/207) \approx .149.$$

(c) The expected return is

$$.1 \times \frac{171}{207} + .18 \times \frac{36}{207} = \frac{2378}{20700}.$$

Q4 Suppose there are n assets which are uncorrelated. The mean return $\bar{r}$ is the same for each asset. The return on asset i has a variance of σ_i^2.

(a) Describe the efficient set.

(b) Find the minim-variance point.

Solution (a) Since each asset has the same average return $\bar{r}$, each portfolio will have the same average return $\bar{r}$ and so the efficent set consists of a single point, $(\sigma, \bar{r})$. It remains to compute σ.
(b) Since $\sum_{i=1}^n w_i \bar{r} = \bar{r}$ is implied by $\sum_{i=1}^n w_i = 1$ we can drop one constraint in our Lagrangean formulation. Our equations then become

$$\sigma_i^2 w_i = \mu \qquad i = 1, 2, \ldots, n.$$

$w_1 + \cdots + w_n = 1$ then implies that

$$\mu = \left(\sum_{j=1}^n \sigma_j^{-2} \right)^{-1}$$

and

$$w_i = \sigma_i^{-2} \left(\sum_{j=1}^n \sigma_j^{-2} \right)^{-1} \qquad i = 1, 2, \ldots, n$$

and

$$\sigma^2 = \sum_{i=1}^n w_i^2 \sigma_i^2 = \left(\sum_{j=1}^n \sigma_j^{-2} \right)^{-1}.$$

Q5 There are 3 assets with data given below:

$$V = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \qquad \bar{r} = \begin{bmatrix} .4 \\ .8 \\ .8 \end{bmatrix}$$

(a) Find the minimum variance portfolio.

(b) Find another efficient portfolio by setting $\lambda = 1, \mu = 0$.

(c) If the risk-free rate is $r_f = .2$, find the efficient portfolio of risky assets.

Solution The equations determining efficient portfolios are

$$\begin{array}{ccccccccccc} 2w_1 & + & w_2 & & & + & .4\lambda & + & \mu & = 0 \\ w_1 & + & 2w_2 & + & w_3 & + & .8\lambda & + & \mu & = 0 \\ & & w_2 & + & 2w_3 & + & .8\lambda & + & \mu & = 0 \\ w_1 & + & w_2 & + & w_3 & & & & & = 1 \\ .4w_1 & + & .8w_2 & + & .8w_3 & & & & & = r \end{array}$$

where r is the target return.
(a) r is not specified i.e. there is no requirement. Thus we drop the 5th equation and put $\lambda = 0$.
By symmetry $w_1 = w_3 = w$ and the equations become

$$\begin{array}{cccccc} 2w & + & w_2 & + & \mu & = 0 \\ 2w & + & 2w_2 & + & \mu & = 0 \\ 2w & + & w_2 & & & = 1 \end{array}$$

So

$$w_1 = \frac{1}{2},\ w_2 = 0,\ w_3 = \frac{1}{2},\ \lambda = 0,\ \mu = -1$$

is the solution.
(b) Now the equations become

$$\begin{array}{cccccccc} 2w_1 & + & w_2 & & & +.4 & = 0 \\ w_1 & + & 2w_2 & + & w_3 & +.8 & = 0 \\ & & +w_2 & + & 2w_3 & +.8 & = 0 \end{array}$$

So

$$w_1 = -.1,\ w_2 = -.2,\ w_3 = -.3,\ \lambda = 1,\ \mu = 0.$$

To get the actual solution we scale this to give

$$w_1 = 1/6,\ w_2 = 1/3,\ w_3 = 1/2.$$

(c) Following the argument in Section 6.9, we solve the following equations (6.10):

$$\sum_{i=1}^{n} \sigma_{k,i} v_i = \bar{r}_k - r_f \qquad k = 1, 2, , \ldots, n$$

$$
\begin{array}{ccccccc}
2v_1 & + & v_2 & & & = .2 \\
v_1 & + & 2v_2 & + & v_3 & = .6 \\
 & & v_2 & + & 2v_3 & = .6
\end{array}
$$

Thus

$$v_1 = .2,\ v_2 = -.2,\ v_3 = -.4.$$

Then we put

$$w_i = \frac{v_i}{\sum_{k=1}^n v_k}$$

yielding

$$w_1 = .5,\ w_2 = -.5,\ w_3 = 1.$$

Q6 Solve the following quadratic programming problem:
Minimise

$$(x_1 + x_2)^2 + (x_1 - x_3)^2 + x_3^2.$$

Subject to

$$x_1 + x_2 + x_3 = 1 \text{ and } x_1, x_2, x_3 \geq 0.$$

Solution

Minimize $(x_1+x_2)^2 + (x_1-x_3)^2 + x_3^2$

Subject to

$$x_1+x_2+x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

KKT conditions:

$$4x_1 + 2x_2 - 2x_3 - u - v_1 = 0$$

$$2x_1 + 2x_2 - u - v_2 = 0$$

$$-2x_1 + 4x_3 - u - v_3 = 0$$

$$x_1, x_2, x_3, v_1, v_2, v_3 \geq 0$$

$$x_1 v_1 = x_2 v_2 = x_3 v_3 = 0.$$

BV	x_1	x_2	x_3	u	v_1	v_2	v_3	RHS
ζ	5	5	3	-3	-1	-1	-1	1
ξ_1	4	2	-2	-1	-1			0
ξ_2	2	2		-1		-1		0
ξ_3	-2		4	-1			-1	0
ξ_4	1	1	1					1

BV	x_1	x_2	x_3	u	v_1	v_2	v_3	RHS
ζ		$\frac{5}{2}$	$\frac{11}{2}$	$-\frac{7}{4}$	$\frac{1}{4}$	-1	-1	1
x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$			0
ξ_2		1	1	$-\frac{1}{2}$	$\frac{1}{2}$	-1		0
ξ_3		1	3	$-\frac{3}{2}$	$-\frac{1}{2}$		-1	0
ξ_4		$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{4}$	$\frac{1}{4}$			1

BV	x_1	x_2	x_3	u	v_1	v_2	v_3	RHS
ζ			3	$-\frac{1}{2}$	-1	$\frac{3}{2}$	-1	1
x_1	1		-1		$-\frac{1}{2}$	$\frac{1}{2}$		0
x_2		1	1	$-\frac{1}{2}$	$\frac{1}{2}$	-1		0
ξ_3			2	-1	-1	1	-1	0
ξ_4			1	$\frac{1}{2}$		$\frac{1}{2}$		1

BV	x_1	x_2	x_3	u	v_1	v_2	v_3	RHS
ζ				1	$-\frac{1}{2}$		$\frac{1}{2}$	1
x_1	1			$-\frac{1}{2}$	-1	1	$-\frac{1}{2}$	0
x_2		1			1	$-\frac{3}{2}$	$\frac{1}{2}$	0
x_3			1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0
ξ_4				1	$\frac{1}{2}$		$\frac{1}{2}$	1

BV	x_1	x_2	x_3	u	v_1	v_2	v_3	RHS
ζ								
x_1	1				$-\frac{3}{4}$	1	$-\frac{1}{4}$	$\frac{1}{2}$
x_2		1			1	$-\frac{3}{2}$	$\frac{1}{2}$	0
x_3			1		$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{2}$
u				1	$\frac{1}{2}$		$\frac{1}{2}$	1

Optimal solution

$x_1 = \frac{1}{2}$, $x_2 = 0$, $x_3 = \frac{1}{2}$.