Balanced Matroids

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Abstract

We introduce the notion of "balance", and say that a matroid is balanced if the matroid and all its minors satisfy the property that, for a randomly chosen basis, the presence of an element can only make any other element less likely.

We establish strong expansion properties for the bases-exchange graph of balanced matroids; consequently, the set of bases of a balanced matroid can be sampled and approximately counted using rapidly mixing Markov chains. Specific classes for which balance is known to hold include graphic and regular matroids.

1 Introduction and Preliminaries

Let S be a finite ground-set, and let $\mathcal{B}$ be a collection of subsets of S. Following standard terminology, the set $\mathcal{B}$ is said to form the collection of *bases* of a *matroid* $\mathcal{M}(S,\mathcal{B})$ if and only if (i) all sets in $\mathcal{B}$ have the same cardinality (called the *rank* of the matroid), and (ii) for any pair of bases B_1 and B_2 the following exchange property holds: for all $e \in B_1$ there exists an $f \in B_2$ such that $B_1 \setminus \{e\} \cup \{f\}$ is in $\mathcal{B}$. A main example is that of *graphic matroids*, whose ground-set is the set of edges of a given graph and whose bases are the spanning trees of the graph. Another example is that of *vectorial matroids*, whose ground-set is a set of vectors over some field and whose bases are maximum cardinality linearly independent subsets of the set of vectors. A subclass of vectorial matroids which will be of interest in this paper is that of *regular matroids*: they are vectorial over every field. All graphic matroids are regular [28].

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The *bases-exchange graph* of a matroid $\mathcal{M}$, henceforth denoted by $G(\mathcal{M})$, was introduced by Edmonds [10] as the graph whose vertex-set is the collection of bases of the matroid, with two bases B_1 and B_2 connected by an edge if and only if B_2 can be obtained from B_1 by the fundamental operation of removing and adding one ground-set element: $B_2 = B_1 \setminus \{e\} \cup \{f\}$. The bases-exchange graph has been studied before in various combinatorial contexts [12, 20]; here we focus on expansion related properties of $G(\mathcal{M})$.

If B is a basis chosen uniformly at random from $\mathcal{B}$, and e is an element of S, let e denote the event $e \in B$ indicating that e is in the chosen basis. The matroid $\mathcal{M}(S,\mathcal{B})$ is said to satisfy the *negative correlation* property if the inequality

$$\Pr[ef] \le \Pr[e]\Pr[f]$$

holds for all pairs of distinct elements e, f in S. Realize that negative correlation is equivalent to $\Pr[e|f] \le \Pr[e]$, thus expressing the intuitive fact that the presence of an element f can only make another element e less likely; the negative correlation property has been studied before in [5, 23]. Regular and in particular graphic matroids are known to be negatively correlated.

Here we strengthen the notion of negative correlation to that of "balance": We say that a matroid $\mathcal{M}(S,\mathcal{B})$ is *balanced* if all its minors, including $\mathcal{M}$ itself, satisfy the negative correlation property. (A *minor* of a matroid is obtained by repeatedly performing the operation of choosing an element $e \in S$ and then selecting either those bases that contain e or those that do not. In the case of graphic matroids, this operation corresponds to contraction or deletion of selected edges from the graph.)

We first establish (Theorem 3.4) that the bases-exchange graph $G(\mathcal{M})$ of any balanced matroid $\mathcal{M}$ has *cutset expansion* 1, i.e., for any bipartition of the vertices of $G(\mathcal{M})$, the number of edges incident to both partition classes is at least as large as the size of the smaller partition class. Such strong expansion properties had been conjectured by Mihail and Vazirani [19], in fact for significantly wider classes of graphs. The core of our proof here is to show that the negative correlation property for two elements e and f implies negative correlation between an element e and an arbitrary monotone property m on $S \setminus \{e\}$,

$$\Pr[em] \leq \Pr[e]\Pr[m] \quad ,$$

which implies an enforcement of vertex neighborhood analogous to bipartite expansion for certain subgraphs on $G(\mathcal{M})$ (formalized as a *"ratios enforcement condition"* in [18]), which finally implies expansion for $G(\mathcal{M})$ (this last step was observed in [18]). Previous arguments for expansion of combinatorial graphs involved using elements of the state-space to encode paths and bound path congestion [6, 14, 19], the discrete version of isoperimetries from differential geometry [9, 16], and coupling arguments [2, 4].

The algorithmic significance of the expansion of the bases-exchange graph derives from a sequence of well known ideas, see for example [1, 3, 9, 14, 15, 19, 25], as follows:

• Consider the *natural random walk* X_t, $t = 0, 1, \ldots$ on $G(\mathcal{M})$: If X_t is the state (basis) at time t, then with probability one half $X_{t+1} = X_t$, and with probability one half X_{t+1} is determined as follows: Choose e from X_t and f from S uniformly at random and, if $X' = X_t \setminus \{e\} \cup \{f\} \in \mathcal{B}$ then $X_{t+1} = X'$, otherwise $X_{t+1} = X_t$. It is easy to see that the Markov chain X_t can be simulated efficiently (given, say, an independence oracle), and that it converges to the uniform distribution over $\mathcal{B}$ (it is symmetric). Most importantly, the expansion of $G(\mathcal{M})$ suggests that X_t has large *conductance*, and hence possesses the *rapid mixing property* which amounts, roughly, to X_t approaching its stationary distribution arbitrarily close for $t = \text{poly-log}(|\mathcal{B}|)$. Therefore, the natural random walk on $G(\mathcal{M})$ can be used as an efficient almost uniform sampling scheme for the set of bases of any balanced matroid.

• In turn, efficient almost uniform sampling for self-reducible combinatorial populations yields efficient Monte-Carlo approximation schemes for their size, and, in view of the fact of self-reducibility for balanced matroids, we obtain an efficient randomized algorithm to approximately count the number of bases of any balanced matroid. In general, exact counting of bases of a matroid is #P-complete [27].

• Our results thus show that the set of bases of a matroid can be efficiently sampled and approximately counted if balance can be established for the matroid. Balance is known to hold for graphic and more generally regular matroids; some counter-examples to balance are also known.

We further show that balance can be used to define random paths between any pair of bases on the bases-exchange graph, such that the length of these paths and the path congestion through any specific edge can be bounded (Section 4). This gives an alternative proof of cutset expansion. The technique used here to bound path congestion is derived from the ratios enforcement condition by means of establishing the existence of certain *fractional matchings*, and these in turn *control the flow of paths* through any edge of the bases-exchange graph. The only previously known method to bound path congestion was Jerrum and Sinclair's argument of using the state-space to encode paths [14]; such arguments have been successful only for combinatorial graphs with strong symmetry properties, e.g. matching graphs [6, 14, 19]. Our path arguments here are of different type; in fact, matroid related graphs do not appear to possess symmetry properties. Furthermore, by extending a technique of Diaconis and Strook [7] and Mihail [17] we show that an analysis that uses paths rather than conductance can yield a sharper convergence rate for the natural random walk as well as for a *modified version of the natural random walk*. The algorithmic significance of this improvement is as follows:

• For arbitrary balanced matroids, where n is the rank, m is the cardinality of the ground-set, and ϵ is a bound on total variation distance, we obtain a sampling scheme based on the natural random walk with $O((n \log m + \log \epsilon^{-1}) mn^2)$ convergence rate (Theorem 5.1). In particular, this result applies to graphic and regular matroids and significantly improves upon previously known bounds. For graphic matroids, for the natural random walk, Broder's coupling and conductance arguments yielded, roughly, $O((n \log m + \log \epsilon^{-1}) m^2 n^4)$ convergence rate [4]. For regular matroids, Dyer and Frieze's geometric arguments yielded $O((n \log m + \log \epsilon^{-1}) m^4 n^4)$ convergence rate [8].

- We analyze a modification of the natural random walk and show $O((n \log m + \log \epsilon^{-1}) n^3)$ convergence rate (Theorem 5.2). This bound is the first known bound for matroids that remains polynomial even for an exponentially large ground-set.

- For regular matroids we need $O(mn)$ time to implement each step of the modified random walk (details of this implementation are tedious but straightforward and are left for the full paper), which results in $O((n \log m + \log \epsilon^{-1}) mn^4)$ running time. There is an alternative sampling scheme based on Kirkhoff's tree matrix theorem [28] (involving evaluation of determinants and arithmetic modulo primes) with running time $O(mn^4 \log m)$. However, even though our scheme is worse by a factor of n, it is very simple conceptually and much easier to implement (furthermore our analysis of the running time is not necessarily tight).

- For graphic matroids we need $O(\sqrt{m})$ time to implement each step of the modified random walk, along the lines of Frederickson [11], which results in $O((n \log m + \log \epsilon^{-1}) \sqrt{m} n^3)$ running time. For the natural random walk, each step can be implemented in $O(\log n)$ time [11]. There is an alternative sampling scheme due to Aldous [2] and Broder [4] with $E(C)$ running time, where C is the cover time of the underlying graph. That scheme thus runs in expected time $O(nm)$ and is clearly faster than the one presented here. However, when we introduce a large number of parallel edges in the graph the expectation of the cover time blows up analogously.

- For the latter case of graphs with a large number of parallel edges we obtain a sampling scheme with $O((n \log m' + \log \epsilon^{-1}) \sqrt{m'} n^3)$ running time, where m' is the number of edges when parallel edges are counted only once; this follows from Theorem 5.3. (More generally, the introduction of parallel elements for a balanced matroid can significantly increase the convergence rate of the natural random walk, but leaves the bounds for the modified random walk unchanged.) The alternative scheme based on Kirkhoff's tree matrix theorem is both less efficient, $O(m' n^4 \log m)$, and more complicated to implement. The ability to introduce a large number of parallel edges without any increase in the running time makes it possible, for example, to output a spanning tree in a graph where each edge has an associated failure probability (such probabilities can be represented by introducing a number of parallel edges proportional to the probability of the edge remaining present), so that each spanning tree is output with probability proportional to that of its presence in a random configuration of the graph.

We give a combinatorial proof of balance for regular matroids, exhibiting a structure that can be used to significantly reduce the convergence rates for certain regular matroids. Previous proofs were of algebraic nature; e.g. see [5] for the special case of graphic matroids.

The remainder of the paper is organized as follows. Section 2 is concerned with classes of matroids that satisfy or fail to satisfy balance: We give a combinatorial proof of balance for regular matroids, and we discuss counter-examples to balance. Section 3 exploits balance to satisfy a certain ratios condition and establish expansion. Section 4 uses the existence of certain fractional matchings derived from the ratios condition to construct paths in the bases-exchange graph; these paths are then used in Section 5 to bound the convergence rates of two sampling schemes for balanced matroids. Section 6 introduces a decomposition property that follows in certain cases from the construction in Section 2 and implies then tighter bounds on convergence rates. Section 7 summarizes the context of this work and outlines directions for further research.

2 Achieving Balance

Consider a minor-closed family of matroids that satisfy the negative correlation property. Then all matroids in the family are balanced as well. For example, graphic matroids whose bases are the edge sets of spanning trees in a given graph are balanced. More generally, all regular matroids are negatively correlated, and the fact that their minors are also regular implies that all regular matroids are balanced. We give here a new combinatorial proof of the fact that regular matroids are balanced.

We refer to the circuits and cocircuits of a matroid as "cycles" and "cuts", by analogy with the case of graphic matroids. A *cycle* is thus a minimal set which cannot be augmented to a basis, while a *cut* is a minimal set whose complement does not contain a basis.

The regular matroids are known to be the *orientable binary matroids*, defined by the following property [28]. It is possible to assign (1) values $C(g)$ for each cycle C and each element g, so that $C(g) = \pm 1$ if $g \in C$ and $C(g) = 0$ otherwise; and (2) values $D(g)$ for each cut D and each element g, so that $D(g) = \pm 1$ if $g \in D$ and $D(g) = 0$ otherwise; most importantly,

$$\sum_g C(g)D(g) = 0$$

for all cycles C and cuts D. Intuitively, in the case of graphic matroids, assign a conventional direction to all edges in the underlying graph, then traverse each cycle C in one of the two possible directions, setting $C(g) = \pm 1$ for each edge traversed depending on whether the edge is traversed in the conventional direction; the edges of each cut D separate a set A of vertices in the graph from its complement $\overline{A}$, so traverse all edges in the cut from A to $\overline{A}$, setting $D(g) = \pm 1$ depending on whether the edge is traversed in the conventional direction. Since a cycle will traverse a cut the same number of times from A to $\overline{A}$ as from $\overline{A}$ to A, it follows that the sum over all g of $C(g)D(g)$ is zero.

We refer to sets which can be obtained from bases by removing one element as *near-bases*. Every near-basis N defines a unique cut D_N contained in the complement of N. We refer to sets which can be obtained from bases by adding one element as *unicycles*. Every unicycle U defines a unique cycle C_U contained in U. A useful property relating near-bases and unicycles is the following. If $U = N \cup \{e, f\}$, then

$$D_N(e)D_N(f) = -C_U(e)C_U(f).$$

To see this, note that the only elements that could be both in D_N and in C_U are e and f, so that by the zero sum property we have $C_U(e)D_N(e) + C_U(f)D_N(f) = 0$; the above equality follows then if all four quantities involved are zero, while if one of them is non-zero, say $D_N(e) \neq 0$, then $N \cup \{e\}$ is a basis, so $U \backslash \{f\}$ is a basis and $C_U(f) \neq 0$, giving $D_N(f)/D_N(e) = -C_U(e)/C_U(f)$.

We can now define an important quantity. If $e \neq f$, we let

$$\Delta_{ef} = \sum_N D_N(e)D_N(f) = -\sum_U C_U(e)C_U(f).$$

The equality of the two expressions follows from the fact that non-zero terms in the sums arise only with pairs N, U such that $U = N \cup \{e, f\}$. For the theorem below, it will be convenient to select a specific element e and require that $D(e) = -C(e) = 1$ for all cycles C and cuts D containing it. This condition can easily be enforced by changing the signs of all elements of chosen cycles or cuts, without violating the conditions on $C(g), D(g)$, and without affecting the value of Δ_{ef}. With this condition, if e belongs to the cycles and cuts involved and $U = N \cup \{e, f\}$, then the above equations become simply

$$\begin{aligned} \sum_{g \neq e} C(g)D(g) &= 1 \\ D_N(f) &= C_U(f) \\ \Delta_{ef} &= \sum_{N: e \in D_N} D_N(f) = \sum_{U: e \in C_U} C_U(f). \end{aligned}$$

Intuitively, for graphic matroids, the quantity Δ_{ef} measures whether cycles containing e, f arising from unicycles tend to traverse e and f in the same or in opposite directions, or equivalently, whether cuts containing e, f arising from near-bases tend to be traversed by e and f in the same or in opposite directions. For graphic matroids, we can show that the quantity Δ_{ef} coincides with a quantity from the theory of electrical networks: If a unit resistance is assigned to each edge in the graph, and a current of $|\mathcal{B}|$ enters and leaves at the endpoints of e, then the potential drop between the endpoints of f is Δ_{ef}. The result below has been shown before for graphs with Δ_{ef} defined as a potential drop in [5]. We use indices on $\mathcal{B}$ to indicate subsets of $\mathcal{B}$ satisfying certain conditions concerning the presence or absence of certain elements in the chosen bases.

Theorem 2.1 *The bases of a regular matroid satisfy* $|\mathcal{B}| \cdot |\mathcal{B}_{ef}| = |\mathcal{B}_e| \cdot |\mathcal{B}_f| - \Delta_{ef}^2$.

Proof. From pairs $(B, B') \in \mathcal{B}_{\overline{e}} \times \mathcal{B}_{ef}$, we obtain pairs $(B'', B''') \in \mathcal{B}_e \times \mathcal{B}_{\overline{e}f}$, by means of an exchange involving e and an element $g \neq e$. More specifically, we let $B'' = B \cup \{e\} \backslash \{g\}$, and $B''' = B' \backslash \{e\} \cup \{g\}$, and assign a *weight* to this exchange equal to $C_{B \cup \{e\}}(g) D_{B' \backslash \{e\}}(g)$. If this weight is non-zero, then the resulting B'' and B''' are indeed bases in $\mathcal{B}_e$ and $\mathcal{B}_{\overline{e}f}$ respectively, and a non-zero weight cannot occur here for $g = f$ since $D_{B' \backslash \{e\}}(f) = 0$.

The above weight can also be expressed as $D_{B'' \backslash \{e\}}(g) C_{B''' \cup \{e\}}(g)$. From the point of view of a pair $(B'', B''') \in \mathcal{B}_e \times \mathcal{B}_{\overline{e}f}$, if this weight is nonzero for some $g \neq e, f$, then the reverse exchange $B = B''\backslash\{e\} \cup \{g\}$ and $B' = B''' \cup \{e\}\backslash\{g\}$ gives back a pair $(B, B') \in \mathcal{B}_{\overline{e}} \times \mathcal{B}_{ef}$. Therefore

$$
\begin{aligned}
& |\mathcal{B}_e| \cdot |\mathcal{B}_{\overline{e}f}| \\
= & \sum_{(B'',B''') \in \mathcal{B}_e \times \mathcal{B}_{\overline{e}f}} \big(\sum_{g \neq e} D_{B''\backslash\{e\}}(g) C_{B'''\cup\{e\}}(g)\big) \\
= & \sum_{(B'',B''') \in \mathcal{B}_e \times \mathcal{B}_{\overline{e}f}} \big(\sum_{g \neq e,f} D_{B''\backslash\{e\}}(g) C_{B'''\cup\{e\}}(g)\big) \\
& + \sum_{(B'',B''') \in \mathcal{B}_e \times \mathcal{B}_{\overline{e}}} D_{B''\backslash\{e\}}(f) C_{B'''\cup\{e\}}(f) \\
= & \sum_{(B,B') \in \mathcal{B}_{\overline{e}} \times \mathcal{B}_{ef}} \big(\sum_{g \neq e} C_{B\cup\{e\}}(g) D_{B'\backslash\{e\}}(g)\big) \\
& + \big(\sum_{B'' \in \mathcal{B}_e} D_{B''\backslash\{e\}}(f)\big)\big(\sum_{B''' \in \mathcal{B}_{\overline{e}}} C_{B'''\cup\{e\}}(f)\big) \\
= & |\mathcal{B}_{\overline{e}}| \cdot |\mathcal{B}_{ef}| + \Delta_{ef}^2
\end{aligned}
$$

and the theorem follows. □

The theorem immediately implies $\Pr[ef] \leq \Pr[e]\Pr[f]$, so negative correlation and hence balance follow for regular matroids.

Regular matroids are a subclass of the binary matroids (i.e., vectorial over $GF[2]$). There is a binary matroid S_8 that does not satisfy the negative correlation property [23]; it is known that all binary matroids not containing S_8 as a minor are balanced [22]. Some additional matroids that violate the negative correlation property have recently been found [24] and shown to constitute counter-examples to negative correlation for truncations of the graphic (forests of fixed cardinality) and for truncations of the dual of the graphic (connected spanning subgraphs of fixed cardinality) [26], as well as for transversals. For instance, if we consider the graph consisting of a path of length 5 with each edge replaced by two parallel edges, add an edge e joining the endpoints of the path, and add a self-loop f anywhere, then for connected spanning subgraphs with 6 edges we have $\Pr[ef] = \frac{5}{24} > \frac{17}{24} \cdot \frac{7}{24} = \Pr[e]\Pr[f]$. The matroid just described as a truncation of the dual of the graphic can also be shown to be a truncation of the graphic and a transversal matroid as well.

3 From Balance to Ratios, Expansion, and Fractional Matchings

In this section we derive expansion for balanced matroids. The proof is inductive on the rank and, in an intermediate step, relies on a certain enforcement of ratios analogous to bipartite expansion. More specifically, for a matroid $\mathcal{M}(S, \mathcal{B})$ with bases-exchange graph $G(\mathcal{M})$, let $G_e(\mathcal{M}) = G(\mathcal{B}_e, \mathcal{B}_{\overline{e}}, E)$ denote the bipartite subgraph of $G(\mathcal{M})$ where edges that correspond to exchanges not involving a specific element e are omitted. Let further Γ_e denote vertex neighborhood in $G_e(\mathcal{M})$. The bases-exchange graph $G(\mathcal{M})$ is said to *enforces ratios* if for every element $e \in S$ the following holds [18]:

$$
\begin{aligned}
\forall \mathcal{A} \subseteq \mathcal{B}_e \qquad & \frac{|\Gamma_e(\mathcal{A})|}{|\mathcal{B}_{\overline{e}}|} \geq \frac{|\mathcal{A}|}{|\mathcal{B}_e|} \\
\forall \mathcal{A} \subseteq \mathcal{B}_{\overline{e}} \qquad & \frac{|\Gamma_e(\mathcal{A})|}{|\mathcal{B}_e|} \geq \frac{|\mathcal{A}|}{|\mathcal{B}_{\overline{e}}|} \quad .
\end{aligned}
$$

Lemma 3.1 *For every balanced matroid $\mathcal{M}(S, \mathcal{B})$, the bases exchange graph $G(\mathcal{M})$ enforces ratios.*

Proof. Let $\mathcal{A} \subseteq \mathcal{B}_e$ and let $m_{\mathcal{A}} = \bigvee_{B \in \mathcal{A}} \bigwedge_{e_i \in B, e_i \neq e} e_i$. Note that the set of bases in $\mathcal{B}_e$ satisfying $m_{\mathcal{A}}$ is precisely the set $\mathcal{A}$, while the set of bases in $\mathcal{B}_{\overline{e}}$ satisfying $m_{\mathcal{A}}$ is precisely the set $\Gamma_e(\mathcal{A})$. Hence the first ratios condition is equivalent to

$$\Pr[m_{\mathcal{A}}|\overline{e}] \geq \Pr[m_{\mathcal{A}}|e] \quad . \tag{1}$$

Analogously, for $\mathcal{A} \subseteq \mathcal{B}_{\overline{e}}$, let $\overline{m_{\mathcal{A}}} = \bigvee_{B \in \mathcal{A}} \bigwedge_{e_i \notin B, e_i \neq e} \overline{e_i}$, and note that the set of bases in $\mathcal{B}_{\overline{e}}$ satisfying $\overline{m_{\mathcal{A}}}$ is the set $\mathcal{A}$, and the set of bases in $\mathcal{B}_e$ satisfying $\overline{m_{\mathcal{A}}}$ is the set $\Gamma_e(\mathcal{A})$. Hence the second ratios condition is equivalent to

$$\Pr[\overline{m_{\mathcal{A}}}|e] \geq \Pr[\overline{m_{\mathcal{A}}}|\overline{e}] \quad . \tag{2}$$

In turn, the last two conditions follow from the lemma below:

Lemma 3.2 (Main Lemma) *For every balanced matroid $\mathcal{M}(S, \mathcal{B})$, any monotone property m over the variables in $S \setminus \{e\}$ is negatively correlated with e:*

$$\Pr[me] \leq \Pr[m]\Pr[e] \quad .$$

Proof. We show equivalently that $\Pr[m|e] \le \Pr[m]$. The reasoning is inductive on the size of the ground-set. The case where $\mathcal{M}$ has rank $n=1$ is easy to verify (and this is also the only point where the monotonicity of m is used). For the inductive step note that

$$\begin{aligned} \Pr[m|e] &= \Pr[f|e]\Pr[m|fe] + \Pr[\overline{f}|e]\Pr[m|\overline{f}e], \\ \text{and } \Pr[m] &= \Pr[f]\Pr[m|f] + \Pr[\overline{f}]\Pr[m|\overline{f}]. \end{aligned}$$

Note further that (i) $\Pr[f|e] \le \Pr[f]$ from the fact that $\mathcal{M}$ is balanced, (ii) $\Pr[m|fe] \le \Pr[m|f]$ by applying the inductive hypothesis on $\mathcal{B}_f$ for the property m with the variable f forced to 1, and (iii) $\Pr[m|\overline{f}e] \le \Pr[m|\overline{f}]$ by applying the inductive hypothesis on $\mathcal{B}_{\overline{f}}$ for the monotone property m with the variable f forced to 0. If, in addition, it was the case that (iv) $\Pr[m|fe] \ge \Pr[m|\overline{f}e]$, then the lemma would follow by averaging principles: $\Pr[m|e] \le \Pr[f]\Pr[m|fe] + \Pr[\overline{f}]\Pr[m|\overline{f}e]$ by (i) and (iv), $\le \Pr[f]\Pr[m|f]+\Pr[\overline{f}]\Pr[m|\overline{f}] = \Pr[m]$ by (ii) and (iii).

We argue that there exists always some element f such that (iv) holds. In particular, note that

$$\sum_{f \ne e} \Pr[f|me] = n-1 = \sum_{f \ne e} \Pr[f|e] \quad .$$

Hence for some f, $\Pr[f|me] \ge \Pr[f|e]$ (with $\Pr[f|e] > 0$), which is equivalent to $\Pr[m|fe] \ge \Pr[m|e]$, which is finally equivalent to $\Pr[m|fe] \ge \Pr[m|\overline{f}e]$. This completes the proof of Lemma 3.2, and Lemma 3.1. □□

Remark: The proof of the lemma can be extended to show that if m_1 and m_2 are two monotone properties over disjoint sets of variables from the ground-set, then $\Pr[m_1 m_2] \le \Pr[m_1]\Pr[m_2]$.

Say that a bipartite graph $G(U, V, E)$ admits a *fractional matching* if there exists an assignment of nonnegative weights to the edges in E such that for each vertex in $u \in U$, the sum of the weights of edges incident on u is $|V|$, and for each vertex in $v \in V$, the sum of the weights of edges incident on v is $|U|$.

Corollary 3.3 *For any balanced matroid $\mathcal{M}(S,\mathcal{B})$, and for every every element $e \in S$, the bipartite graph $G_e(\mathcal{M}) = G(\mathcal{B}_e, \mathcal{B}_{\overline{e}}, E)$ admits a fractional matching.*

Proof. Consider the bipartite graph G^* obtained from G_e by making $|\mathcal{B}_{\overline{e}}|$ copies of each basis in $\mathcal{B}_e$, making $|\mathcal{B}_e|$ copies of each basis in $\mathcal{B}_{\overline{e}}$, and including edges between all copies of each pair of adjacent bases in G_e. Enforcement of ratios for $G(\mathcal{M})$ implies now that G^* satisfies Hall's condition, and hence G^* has a perfect matching P. By identifying the copies of each basis and assigning to each edge of E a weight equal to the number of edges of P that correspond to it, the conditions for a fractional matching are satisfied. □

Theorem 3.4 *For every balanced matroid $\mathcal{M}(S,\mathcal{B})$, the bases-exchange graph of $\mathcal{M}$ has cutset expansion 1.*

Proof. In Section 4 we argue that, by means of fractional matchings, we can define random paths between any pair of bases such that the expected number of paths through any specific edge is at most $|\mathcal{B}|/2$ (Corollary 4.2). Thus for $\mathcal{A} \subseteq \mathcal{B}$, there are $|\mathcal{A}| \cdot |\overline{\mathcal{A}}|$ paths constructed from $\mathcal{A}$ to $\overline{\mathcal{A}}$; each path leaves $\mathcal{A}$ through some edge in the cutset of $\mathcal{A}$: $C(\mathcal{A})$; but, by linearity of expectations and the bound on path congestion, the expected number of paths leaving $\mathcal{A}$ is at most $|C(\mathcal{A})| \cdot |\mathcal{B}|/2$; and therefore $|C(\mathcal{A})| \ge 2|\mathcal{A}| \cdot |\overline{\mathcal{A}}|/|\mathcal{B}| \ge \min(|\mathcal{A}|, |\overline{\mathcal{A}}|)$. □

Remark: An alternative inductive proof that by-passes paths and shows expansion directly from ratios enforcement was obtained previously in [18].

Corollary 3.5 *For any balanced matroid $\mathcal{M}$, the natural random walk on the bases exchange graph $G(\mathcal{M})$ is rapidly mixing: The conductance is $\Phi \ge 1/2mn$, and hence the total variation distance $d(t)$ can be bounded by ϵ in time $t = \Omega(n \log m + \log \epsilon^{-1})m^2 n^2$.*

Proof. The total variation distance $d(t)$ has been bounded from above by $|\mathcal{B}|\left(1 - \Phi^2/2\right)^t$ in [25], where the conductance Φ in the case of symmetric Markov chains is the product of the transition probabilities with the cutset expansion. Hence, from Theorem 3.4 and the definition of the natural random walk, Φ can be bounded by by $1/2mn$. □

4 From Fractional Matchings to Path Congestion

In order to obtain bounds on convergence rates that are tighter than those of Corollary 3.5, we shall use the fractional matchings defined in Corollary 3.3 of the previous section to construct paths joining all pairs of bases in the bases-exchange graph of a balanced matroid, while keeping the path lengths and the number of paths through each basis small.

We thus wish to construct $|\mathcal{B}|^2$ paths in the bases-exchange graph of a balanced matroid $\mathcal{M}(S,\mathcal{B})$, one path for each choice of an origin B and a destination B'. The construction begins with the choice a random permutation $e_1,\ldots,e_m$ of the m elements of S. Consider a specific destination $B' = B_{\dot{e}_1\ldots\dot{e}_m}$, where each $\dot{e}_i$ is either e_i or $\overline{e_i}$, depending on whether e_i is present or absent in B'. Note that the paths with destination B' are initially uniformly arranged over the set of all bases $\mathcal{B}$, in that each basis B has exactly one path with destination B'. We shall ensure that after i steps, the paths with destination B' are uniformly arranged over $\mathcal{B}_i = \mathcal{B}_{\dot{e}_1\ldots\dot{e}_i}$, the set of bases that agree with B' in the first i elements, in the sense that the expected number of paths at each such basis after i steps is the same for all of them. This condition holds initially, when $i = 0$ and $\mathcal{B}_i = \mathcal{B}$. Assume inductively that after i steps, the paths with destination B' are uniformly arranged over $\mathcal{B}_i = \mathcal{B}_{i+1} \cup \mathcal{B}'_{i+1}$, where $\mathcal{B}_{i+1} = \mathcal{B}_{\dot{e}_1\ldots\dot{e}_i\dot{e}_{i+1}}$ and $\mathcal{B}'_{i+1} = \mathcal{B}_{\dot{e}_1\ldots\dot{e}_i\overline{\dot{e}_{i+1}}}$. The paths at $\mathcal{B}_i$ that are already at $\mathcal{B}_{i+1}$ are left there for the $(i+1)$th step. The paths at $\mathcal{B}_i$ that are currently at $\mathcal{B}'_{i+1}$ are sent to $\mathcal{B}_{i+1}$ by means of a fractional matching from $\mathcal{B}'_{i+1}$ to $\mathcal{B}_{i+1}$. Such a fractional matching exists by Corollary 3.3 applied to the minor $\mathcal{M}_{\dot{e}_1\ldots\dot{e}_i}$; the probabilities given to each possible exchange are chosen proportionally to the associated weights in the fractional matching, thus ensuring that after $(i+1)$ steps, the paths with destination $\mathcal{B}'$ are uniformly arranged over $\mathcal{B}_{i+1}$. After m steps, all paths are at $\mathcal{B}_m = \{B'\}$, as desired.

Lemma 4.1 *The expected number of paths leaving a basis $\hat{B}$ in $\mathcal{B}_{\dot{e}_1\ldots\dot{e}_i}$ at step i is $|\mathcal{B}|\Pr[\overline{\dot{e}_i}|\dot{e}_1\ldots\dot{e}_{i-1}]$.*

Proof. The paths leaving a basis $\hat{B}$ in $\mathcal{B}_{\dot{e}_1\ldots\dot{e}_i}$ at step i are those with destination some B' in $\mathcal{B}_{\dot{e}_1\ldots\dot{e}_{i-1}\overline{\dot{e}_i}}$. For each such B', before step i, the $|\mathcal{B}|$ paths destined to it are uniformly arranged over $\mathcal{B}_{\dot{e}_1\ldots\dot{e}_{i-1}}$, so at each of these bases there are $|\mathcal{B}|/|\mathcal{B}_{\dot{e}_1\ldots\dot{e}_{i-1}}|$ paths with destination B', in expectation. Mutliplying this quantity by the number $|\mathcal{B}_{\dot{e}_1\ldots\dot{e}_{i-1}\overline{\dot{e}_i}}|$ of possible destinations B' gives the quantity $|\mathcal{B}|\Pr[\overline{\dot{e}_i}|\dot{e}_1\ldots\dot{e}_{i-1}]$ as required. ▯

The same bound holds for paths entering $\hat{B}$. Note that a particular exchange involving two elements e_i and e_j with $i < j$ can be used only during step i, and that the expected number of paths using such an exchange at step i is at most $|\mathcal{B}|\min(\Pr[e_i|\dot{e}_1\ldots\dot{e}_{i-1}],\Pr[\overline{e_i}|\dot{e}_1\ldots\dot{e}_{i-1}])$.

Corollary 4.2 *The expected number of paths through any edge $(\hat{B},\hat{B}')$ is at most $|\mathcal{B}|/2$.*

In order to bound the expected length of paths and the expected number of paths through a basis, we give the following lemma.

Lemma 4.3 *Given a basis $\hat{B}$, consider a sequence of bases and elements $B_0, e_1, B_1, e_2, B_2, \ldots, e_m, B_m = \hat{B}$, where B_i is chosen so that it agrees with $\hat{B}$ in $e_1,\ldots,e_i$ and each e_i is chosen uniformly from the elements other than $e_1,\ldots,e_{i-1}$. Then the expected number of e_i such that B_{i-1} and $\hat{B}$ differ in e_i is at most $2n$.*

Proof. If e_i is chosen so that B_{i-1} and $\hat{B}$ differ in it, then with probability $1/2$ the element e_i is present in $\hat{B}$, since the symmetric difference of two bases contains the same number of elements from each. Since $\hat{B}$ has only n elements, we expect such a choice to happen at most $2n$ times. ▯

Corollary 4.4 *The expected length of a path from any B to any B' is at most $2n$.*

Proof. Follows from Lemma 4.3, where $B = B_0, B_1, \ldots, \mathcal{B}_m = B' = \hat{B}$ is the path from B to B' and $e_1,\ldots,e_m$ is the random order in which the edges were chosen to be fixed. ▯

Say that a path goes through a basis if this basis is on the path but is not its destination.

Corollary 4.5 *The expected number of paths through a basis $\hat{B} = B_{\dot{e}_1\ldots\dot{e}_m}$ is at most $2n|\mathcal{B}|$.*

Proof. By Lemma 4.1, this expectation is $|\mathcal{B}|E\left[\sum_{i=1}^{m}\Pr[\overline{\dot{e}}_i|\dot{e}_1,\ldots,\dot{e}_{i-1}]\right]$, the expectation being over all permutations in which the e_i's are chosen. Now Lemma 4.3, where B_0 is chosen uniformly from $\mathcal{B}$ and B_i is chosen uniformly from $\mathcal{B}_{\dot{e}_1\ldots\dot{e}_i}$, implies that $E\left[\sum_{i=1}^{m}\Pr[\overline{\dot{e}}_i|\dot{e}_1,\ldots,\dot{e}_{i-1}]\right] \le 2n$. ▯

5 From Path Congestion to Rapid Mixing

Consider the natural random walk on the bases-exchange graph $G(\mathcal{M})$ of a balanced matroid

$\mathcal{M}(S,\mathcal{B})$, as defined in the introduction. In this section we show how the bounds on path congestion yield bounds on the convergence rate of the natural random walk; the bounds derived here are significantly better than those of Corollary 3.5 in terms of conductance. We further consider the modification of the natural random walk which uses both bases and near-bases; for this modified random walk the convergence rate becomes even faster. The modified random walk Y_t is as follows: With probability one half $Y_{t+1} = Y_t$, and with probability one half Y_{t+1} is determined as follows: (i) if Y_t is a basis then choose e from Y_t uniformly at random and $Y_{t+1} = Y_t \setminus \{e\}$; (ii) if Y_t is a near-basis and $D_{Y_t} \subseteq S$ is the cut of Y_t, then choose f from D_{Y_t} uniformly at random and $Y_{t+1} = Y_t \cup \{f\}$. It is easy to verify that when Y_t is stationary, the probability of a basis is $1/2|\mathcal{B}|$, and the probability of a near-basis with cut D is $|D|/2n|\mathcal{B}|$; furthermore the probability that Y_t is on a basis is exactly $1/2$. Hence Y_t is also appropriate to use as a sampling scheme for $\mathcal{B}$.

We proceed to bound convergence rate in terms of path congestion. In particular, for a Markov chain Z_t, $t = 0, 1, \ldots$ on state space S and transition matrix $P' = \frac{1}{2}(P + I)$, let $\vec{\pi}$ denote the stationary distribution of Z_t. Let $w_{ij} = \pi_i p_{ij}$ measure the *ergodic flow* between states i and j, and suppose that the ergodic flow w_{ij} is the same between any two states for which $p_{ij} \neq 0$, so let w denote this ergodic flow. Assuming that random paths have been defined between any pair A and B of states of the Markov chain (so that these paths use edges (i,j) such that $p_{ij} \neq 0$), let l denote an upper bound on the expected length of the path from A to B. Finally, among all paths that have been defined between pairs of states choose one path at random according to the distribution that assigns probability $\pi_A \pi_B$ to the path from state A to state B. Let L be an upper bound on the probability that the chosen path contains a specific edge (i,j). Then the following bound can be obtained for the variation distance $d(t)$ at time t:

$$d(t) \leq \frac{1}{2}\left(|S|\frac{\pi_{\max}}{\pi_{\min}}\right)^{\frac{1}{2}}\left(1 - \frac{w}{2Ll}\right)^{\frac{t}{2}} \quad .$$

The detailed proof of the above bound is left for the full paper; here we give an outline as follows. Let $h_i(t) = \Pr[Z_t = i] - \pi_i$, and we shall use h_i as short for $h_i(t)$. First, it is not hard to check that

$$d(t) = \frac{1}{2}\sum_i |h_i| \leq \frac{1}{2}\left(|S|\pi_{\max}\right)^{\frac{1}{2}}\left(\sum_i \frac{h_i^2}{\pi_i}\right)^{\frac{1}{2}} \quad .$$

Let $d'(t) = \sum_i \frac{h_i^2}{\pi_i}$. Next, using techniques analogous to those that appeared in [17], the following can be derived: First an equation that attributes discrepancy from stationarity to discrepancy of the h_i's at the endpoints of (j,i)'s such that $p_{ji} \neq 0$:

$$d'(t) = \frac{1}{2}\sum_i\sum_j w_{ji}\left(\left(\frac{h_j}{\pi_j}\right)^2 + \left(\frac{h_i}{\pi_i}\right)^2\right) \quad .$$

Next an equation stating that each step of the Markov chain results in an averaging of the discrepancies along the endpoints of (j,i)'s such that $p_{ji} \neq 0$; thus the total discrepancy should decrease:

$$h_i(t+1) = \frac{1}{2}\sum_j w_{ji}\left(\frac{h_j}{\pi_j} + \frac{h_i}{\pi_i}\right) \quad .$$

And finally a bound stating that the averaging of the discrepancies along the edges is more effective (and hence the convergence is more rapid) if it involves (j,i)'s with significantly different discrepancies at the endpoints of j and i:

$$d'(t) - d'(t+1) \geq \frac{w}{4}\sum_{ij:p_{ij}\neq 0}\left(\frac{h_j}{\pi_j} - \frac{h_i}{\pi_i}\right)^2 \quad .$$

Now recall that paths have been defined between any pair A and B of states of the Markov chain (so that these paths use edges (i,j) such that $p_{ij} \neq 0$). Let P_{AB} denote the chosen path from A to B and the event that P_{AB} was chosen, and let $ij \in P_{AB}$ denote the event that edge (i,j) was used in the path from A to B. Let l_{AB} denote the length of the path from A to B, and recall that l denotes an upper bound on the expected length of l_{AB}. Finally, recall that among all paths that have been defined between pairs of states we choose one path at random according to the distribution that assigns probability $\pi_A \pi_B$ to the path from state A to state B. Recall that L is an upper bound on the probability that the chosen path contains a specific edge (i,j). Then along the lines of [7, 21] (except that we are using expected rather than maximum path length here) we get:

$$
\begin{aligned}
& d'(t) - d'(t+1) \\
\geq\ & \frac{w}{4} \sum_{ij:p_{ij}\neq 0} \left(\frac{h_j}{\pi_j} - \frac{h_i}{\pi_i}\right)^2 \frac{\sum_A \sum_B \Pr[ij \in P_{AB}]\pi_A\pi_B}{L} \\
=\ & \frac{w}{4L} \sum_A \sum_B \pi_A\pi_B \sum_{P_{AB}} \Pr[P_{AB}] \sum_{ij\in P_{AB}} \left(\frac{h_j}{\pi_j} - \frac{h_i}{\pi_i}\right)^2 \\
\geq\ & \frac{w}{4L} \sum_A \sum_B \pi_A\pi_B \sum_{P_{AB}} \Pr[P_{AB}] \frac{1}{l_{AB}} \left(\frac{h_A}{\pi_A} - \frac{h_B}{\pi_B}\right)^2 \\
=\ & \frac{w}{4L} \sum_A \sum_B \pi_A\pi_B \left(\frac{h_A}{\pi_A} - \frac{h_B}{\pi_B}\right)^2 E\left(\frac{1}{l_{AB}}\right) \\
\geq\ & \frac{w}{4Ll} \sum_A \sum_B \pi_A\pi_B \left(\frac{h_A}{\pi_A} - \frac{h_B}{\pi_B}\right)^2 \\
=\ & \frac{w}{2Ll} \sum_i \frac{h_i^2}{\pi_i} = \frac{w}{2Ll} d'(t) \ .
\end{aligned}
$$

Finally using all the above and noting that $d'(0) \leq 1/\pi_{\min}$, we get the desired bound on the variation distance.

For the natural random walk on any balanced matroid $\mathcal{M}(S, \mathcal{B})$ we have $w = 1/|\mathcal{B}|mn$, $l = 2n$ by Corollary 4.4, and $L = 1/2|\mathcal{B}|$ by Corollary 4.2 and the definition of L. Hence

Theorem 5.1 *For the natural random walk on any balanced matroid the total variation distance can be bounded by ϵ in time $t = \Omega(n \log m + \log \epsilon^{-1})n^2 m$.*

For the modified random walk, paths between bases are defined as for the bases-exchange graph in Section 4, except that near-bases are used to simulate the edges of the bases-exchange graph in the obvious way. The expected number of such paths through an edge can be bounded by the expected number of paths through a vertex of the bases-exchange graph; the latter expectation is at most $2n|\mathcal{B}|$ by Corollary 4.5. Paths whose endpoints include near-bases are defined by first choosing a neighboring basis uniformly at random, and then choosing a random path between bases, thus reducing path congestion due to paths with arbitrary endpoints to path congestion due to paths whose endpoints are only bases. From the above observations and the definition of L it is not hard to verify that for the modified random walk $L = (2n+1)/|\mathcal{B}|$. Furthermore, $w = 1/2n|\mathcal{B}|$ and $l = 4n + 2$. Thus

Theorem 5.2 *For the modified random walk on any balanced matroid the total variation distance can be bounded by ϵ in time $t = \Omega(n \log m + \log \epsilon^{-1})n^3$.*

Consider now the case where each element may have a very large number of elements parallel to it. These parallel elements may cause a large increase in the state space size $|S|$. To avoid this increase, collapse all parallel elements, and view the number of parallel elements as a weight on the resulting element after collapsing; the probability of any given basis is now proportional to the product of the weights of elements in it, multiplied by n for bases, and by the sum of the weights of elements in the associated cut for near-bases. This transformation reduces the size of the state space, but may increase the factor $\pi_{\max}/\pi_{\min}$ due to the weight differences. To reduce this quantity, we choose as a starting point for the random walk a basis of maximum weight, so that $\pi_{\min}$ can be replaced by $\pi_{\max-\text{basis}}$ and $\pi_{\max}/\pi_{\max-\text{basis}}$ is at most m/n, where m is the number of elements after parallel elements have been collapsed. Therefore for the modified random walk, unlike the natural random walk, the number of parallel elements does not affect the running time. A basis of maximum weight needed to start the random walk can be found by greedily selecting elements of largest weight while preserving independence.

Theorem 5.3 *The bounds for the modified random walk on any balance matroid remain the same after an arbitrary number of parallel elements are added while maintaining balance.*

The algorithmic significance of Theorems 5.1, 5.2, and 5.3 was pointed out in the introduction.

6 Better Path Congestion, Decomposition, Symmetry, Series Parallel

In the graph consisting of bases and near-bases for a balanced matroid, we have constructed paths between each pair of bases, and bounded the load on edges using a bound of about $n|\mathcal{B}|$ on the load of basis vertices. Since each basis has n incident edges, one for each element of the basis, one might expect the load on each edge to be bounded by $|\mathcal{B}|$. If a bound of $\alpha|\mathcal{B}|$ holds for some α, the n^3 factor of convergence rate for the modified random walk can be reduced to

αn^2. We shall now examine conditions that can yield such bounds.

Consider the bipartite graph with edges joining bases and near-bases. Given an element e, one can find a fractional matching between all bases, each with weight $|\mathcal{B}_e|$, and all near-bases $\mathcal{N}_e$ such that e belongs to their associated cut, each with weight $|\mathcal{B}|$. To see this, note that the bases in $\mathcal{B}_e$ can be matched to the near-bases in $\mathcal{N}_e$, simply by removing e from each of them. The weight assigned to this edge is $|\mathcal{B}_e|$, satisfying the weight constraint for each basis in $\mathcal{B}_e$, and leaving at each near-basis in $\mathcal{N}_e$ a weight of $|\mathcal{B}_{\bar{e}}|$ to be satisfied using the edges joining $\mathcal{B}_{\bar{e}}$ to $\mathcal{N}_e$. If we view each near-basis in $\mathcal{N}_e$ as representing a corresponding basis in $\mathcal{B}_e$, then this bipartite graph is simply the bases-exchange graph associated with e, and we can infer that a fractional matching giving weight $|\mathcal{B}_e|$ to each basis in $\mathcal{B}_{\bar{e}}$ and weight $|\mathcal{B}_{\bar{e}}|$ to each near-basis in $\mathcal{N}_e$ does exist.

Given a near-basis N in $\mathcal{N}_e$ and an element $f \neq e$ that belongs to the cut of N, denote by w_{efN} the weight assigned to the edge joining N to the basis $N \cup \{f\}$ by the above fractional matching for e. For convenience, if $U = N \cup \{e, f\}$, we write $w_{efU} = w_{efN}$. For all other choices of f,N,U, we set $w_{efN} = 0$, $w_{efU} = 0$. The definition of fractional matchings requires $w_{efN} \geq 0$, and in addition, the two conservation equations

$$\sum_f w_{efN} = |\mathcal{B}_{\bar{e}}| \text{ for given } e \text{ and } N \text{ with } e \in D_N,$$

$$\sum_f w_{efU} = |\mathcal{B}_e| \text{ for given } e \text{ and } U \text{ with } e \in C_U.$$

We have thus constructed m fractional matchings from $\mathcal{B}$ to $\mathcal{N}$, one for each choice of e. Suppose that we now add these m fractional matchings, by adding, for each basis, each near-basis, and each edge, its weight from each of the fractional matchings. The resulting weight for each basis is $\sum_e |\mathcal{B}_e| = n|\mathcal{B}|$, and for each near-basis N it is $\sum_{e \in D_N} |\mathcal{B}| = |D_N| \cdot |\mathcal{B}|$. In both cases, the weight is equal to the degree of the basis or near-basis times $|\mathcal{B}|$. Note that there is a very simple fractional matching satisfying these weight constraints, namely the *uniform fractional matching* that assigns weight $|\mathcal{B}|$ to each edge joining a basis and a near-basis. We say that a balanced matroid has the *exact decomposition property* if the uniform fractional matching can indeed be obtained as the sum of m fractional matchings, one for each element e. In terms of weights, this means that the weights assigned to a particular edge joining N and $N \cup \{f\}$ must add to $|\mathcal{B}|$, and since the weight of this edge for the fractional matching with $e = f$ is $|\mathcal{B}_f|$, we must have

$$\sum_e w_{efN} = |\mathcal{B}_{\bar{f}}| \text{ for given } f \text{ and } N \text{ with } f \in D_N.$$

Note that this property must hold if *symmetry* holds, namely, if $w_{efN} = w_{feN}$ for all e, f, N. If symmetry holds, then exact decomposition follows from the first conservation equation. One can also consider the weaker α-*decomposition* property, which only requires

$$\sum_e w_{efN} \leq \alpha|\mathcal{B}|$$

for some α. Note that this property always holds for $\alpha = n$, since the sum weight of the basis $N \cup \{f\}$ is $n|\mathcal{B}|$.

Lemma 6.1 *If a balanced matroid and all its minors satisfy the α-decomposition property, then its path congestion is at most $(\alpha + 2)|\mathcal{B}|$ and the n^3 factor of the convergence rate for the modified random walk reduces to αn^2.*

Proof. Consider the first step of all paths. Let e be the element chosen for the first step, and consider an edge joining $N \cup \{f\}$ to N. If $e = f$, then the load of this edge is $|\mathcal{B}_{\bar{f}}|$. If $e \neq f$, then the load is determined by the fractional matching, and it is precisely w_{efN}. Since e is chosen uniformly at random from all m elements, the expected load is

$$\frac{1}{m}(|\mathcal{B}_{\bar{f}}| + \sum_e w_{efN}) \leq \frac{\alpha+1}{m}|\mathcal{B}|.$$

At the $(i+1)$th step of the paths, if f is one of the i elements $e_1, e_2, \ldots, e_i$ used for the first i steps, then the load for the edge under consideration is zero. This case occurs with probability i/m. Otherwise, the edge joining $N \cup \{f\}$ to N will be involved in the fractional matching for some $e = e_{i+1}$ after the chosen $e_1, e_2, \ldots, e_i$ have been contracted or deleted, depending on whether they occur in N or not. This minor has only $m - i$ elements, so the expected load is now bounded by $\frac{\alpha+1}{m-i}|\mathcal{B}|$ in this case. Since this second case occurs with probability $(m-i)/m$, the expected load for this step is again bounded by $\frac{\alpha+1}{m}|\mathcal{B}|$. Adding this quantity over all m steps of the paths (plus the initial load assignment from near-bases to bases), we obtain a bound of $(\alpha+2)|\mathcal{B}|$ on the path congestion, thus reducing the n^3 factor for the convergence rate to αn^2. □

The main obstacle in obtaining bounds for the values of α achievable for α-decompositions is the fact that the fractional matchings are in general not explicitly known. We now give a case where such an explicit construction can be obtained.

Theorem 6.2 *If a regular matroid satisfies $D(e)D(f)\Delta_{ef} \geq 0$ for all cuts D and elements e, f, then symmetry and exact decomposition hold. The regular matroids such that exact decomposisition (as well as symmetry) holds for the matroid and all its minors are precisely the graphic matroids on graphs whose biconnected components are either K_4 or a series-parallel graph.*

Proof. We set $w_{efN} = D_N(e)D_N(f)\Delta_{ef}$. These quantities are nonnegative by assumption, and satisfy the conservation laws: If $e \in D_N$, then

$$\begin{aligned}\sum_f w_{efN} &= -\sum_{f\neq e} D_N(e)D_N(f)\sum_U C_U(e)C_U(f)\\ &= \sum_U (D_N(e)C_U(e))^2 = \sum_{U:e\in C_U} 1 = |\mathcal{B}_{\overline{e}}|.\end{aligned}$$

Similarly, if $e \in C_U$, then $\sum_f w_{efU} = |\mathcal{B}_e|$. Furthermore, the symmetry property $w_{efN} = w_{feN}$ holds by the definition of Δ_{ef}, and thus exact decomposition holds as well.

There are two dual graphic matroids that do not satisfy symmetry or exact decomposition, namely K_4^1 and K_4^2, obtained from K_4 by replacing a single edge with either (1) two parallel edges, or (2) two edges in series. If both are excluded as minors, then the duals of K_5 and $K_{3,3}$ are also excluded, so the regular matroid must be graphic [28]; furthermore, a biconnected component containing K_4 as a minor must then in fact be K_4, and a biconnected component has no K_4 as minor if and only if it is a series-parallel graph. In a series-parallel graph, one cannot find two cycles through two given edges e, f that traverse e in the same direction but traverse f in opposite directions; otherwise, the two cycles would give a K_4 minor. It follows that $C(e)C(f)$ cannot be nonzero and have different signs for two different cycles C. Therefore $D_N(e)D_N(f)\Delta_{ef} = -C_U(e)C_U(f)\Delta_{ef} \geq 0$, satisfying the conditions for symmetry and exact decomposition. In a clique K_n, one has $\Delta_{ef} = 0$ if e, f do not share any vertices, because there is an isomorphism of K_n that leaves e fixed but swaps the endpoints of f. If e shares an endpoint with f, then the direction used by a cycle through e, f in traversing e determines the direction used in traversing f, so symmetry and exact decomposition hold as before for series-parallel graphs. They hold in particular for K_4, and since all its minors are series-parallel graphs, the two properties hold for the minors as well. □

Remark: The above is not a complete characterization of the balanced binary matroids that have an exact decomposition, since the Fano matroid, an excluded minor for regular matroids, satisfies symmetry (setting $w_{efN} = 4, 6$ depending on whether both or only one of the elements of N form a basis with $\{e, f\}$).

We do not know whether an α constant exists such that the α-decomposition property holds for all balanced matroids.

7 Conclusions and Open Problems

The following conjecture of Mihail and Vazirani [19] remains unresolved. Define the 1-skeleton of a convex polytope as the graph whose vertex-set is the set of vertices of the polytope and whose edges correspond to 1-dimensional faces (edges) of the polytope. The conjecture states that all 0-1 polytopes are expanding, with cutset expansion 1 (a 0-1 polytope is a polytope whose vertices have 0-1 coordinates). A related graph on 0-1 vertices can be obtained by removing from the complete graph all edges that intersect other edges (two such edges must necessarily meet at their midpoint). For certain structured combinatorial problems, where 0-1 vertices correspond to solutions, the two graphs coincide; this is the case, for example, for perfect matchings of a graph and for bases of matroid [19]. For matroids, this graph is the bases-exchange graph. We have introduced balanced matroids, and shown that balance implies cutset expansion 1, and hence obtained efficient sampling and randomized approximation schemes for bases of balanced matroids. It is worth noting the role that vertex-pairs meeting at their midpoint (as used in the definition of the above combinatorial graph) play in proofs of expansion; see [19] and the proof of Theorem 2.1.

An interesting question is the evaluation of the polynomial $f(x) = \sum_{k\geq 0} a_k x^k$ at some point x, where a_k is the number of spanning sets of size $n+k$. For graphic matroids, the value $f(0)$ gives the number of spanning trees, the value $f(1)$ gives the number of connected spanning subgraphs, and $(1-p)^n p^{m-n} f(\frac{1}{p}-1)$ gives network reliability for failure probability p; the evaluation of the last two quantities is #P-complete [27]. More generally $f(x-1) = T(1,x)$, where T is the Tutte polynomial; dual expressions for independent sets (forests in the case of graphic matroids) instead of spanning sets can be obtained by interchanging the arguments of T [13, 29]. The quantities $f(x)$ are multiplicative under the operation of joining two graphs at a single vertex, a property that does not hold for the individual coefficients a_k; for instance, see the counter-example to balance for spanning connected subgraphs in Section 2.

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