Chapter 8: S_n & Shuffling cards

Stack of n cards $\Leftrightarrow \sigma \in S_n$
hence
Shuffing method $\Leftrightarrow$ rw on S_n

rw on S_n: $\sigma_i \to \sigma_{i+1} = \sigma_i \rho$
ρ chosen wrt μ on S_n
Shuffling method $\Leftrightarrow \mu$.

Note: we can generate a uniform random $\sigma \in S_n$ by starting with id, then define $J_1, J_2, \dots, J_{n-1}$ as uniform random variables, with J_k taking values in $\{k, \dots, n\}$. in order, flip position i with position J_i. Ex 8.1 shows this works.

note: if $\langle \mathrm{supp}(\mu) \rangle = S_n$, chain is irred.
$\mu(id) > 0$, $\Rightarrow$ chain is aperiodic.

Random Transpositions

$$\mu(\sigma) = \begin{cases} 1/n & \sigma = id \\ 2/n^2 & \sigma = ij \\ 0 & o.w. \end{cases}$$

(pick 2 random indices and exchange them)

mixing time?

Upper Bound #1: Coupling

at time t, pick $X_t, Y_t \in [n]$ unif at random
Swap card with value X_t with card in posn Y_t.

do this w/ two decks, with same X_t, Y_t.

trajectories $(\sigma_t), (\sigma_t')$. $\quad q_t := |\{i : \sigma_t(i) \neq \sigma_t'(i)\}|$

(coupling)

what can happen in 1 step?

① $\sigma_t^{-1}(X_t) = \sigma_t'^{-1}(X_t)$ and $\sigma_t Y_t = \sigma_t'(Y_t)$

$\Rightarrow a_{t+1} = a_t$

② $=$ $\neq$

$\Rightarrow a_{t+1} = a_t$

③ $\neq$ $=$

$\Rightarrow a_{t+1} = a_t$ ← make 1 pair, break 1

④ $\neq$ $\neq$

$\Rightarrow a_{t+1} > a_t \Leftrightarrow a_{t+1} \in \{a_t+1, a_t+2, a_t+3\}$

$\tau := \min\{t : a_t = n\}$ (coupling time)

(prop 8.5)

$$E(\tau) \leq \frac{\pi^2}{6} n^2$$

Pf let $\tau = \tau_1 + \tau_2 + \dots + \tau_n$

$\tau_i =$ # transp. btwn $a_t \geq i-1$ and $a_t \geq i$

(some can be 0)

$a_t = i \Rightarrow P(\text{new alignment}) = \frac{(n-i)^2}{n^2}$

so, τ_i geom w/ $p = \frac{(n-i)^2}{n^2}$. hence,

$$E(\tau_{i+1} \mid a_t = i) = n^2/(n-i)^2.$$

So $E(\tau) < \sum_{i=0}^{n-1} n^2/(n-i)^2 < n^2 \sum_{j=1}^{\infty} 1/j^2 = \frac{\pi^2}{6} n^2$ ☐

This gives $t_{mix} = O(n^2)$, but the SST next does much better.

Upper Bound #2: SSTs

pick $L_t, R_t \in [n]$ unif at random.
swap those positions.
at $t=0$, no cards are marked.
at t, mark R_t if • L_t marked OR
• $L_t = R_t$.

This is a SST. (b/c of random insertion)

Lemma 8.9) $E(\tau) = 2n(\log(n) + O(1))$
$\mathrm{Var}\,\tau = O(n^2)$

Pf) let $\tau = \tau_0 + \tau_1 + \cdots + \tau_{n-1}$
$\tau_k :=$ # transps. after k^{th} marking, up to (& including) $k+1$-st

τ_k geom w/ $p = \frac{(k+1)(n-k)}{n^2}$

hence
$$E(\tau) = \sum_{k=0}^{n-1} \frac{n^2}{(k+1)(n-k)}$$
$$= \sum_{k=0}^{n-1} \frac{n^2}{n+1}\left(\frac{1}{k+1} + \frac{1}{n-k}\right)$$
$$= 2n(\log n + O(1))$$

$$\mathrm{Var}\,\tau = \sum_{k=0}^{n-1} \frac{1-\frac{(k+1)(n-k)}{n^2}}{\left(\frac{(k+1)(n-k)}{n^2}\right)^2} < \sum_{k=0}^{n-1} \frac{n^4}{(k+1)^2(n-k)^2}$$
$$= \sum_{0\le k< n/2} \frac{n^4}{(k+1)^2(n-k)^2} + \sum_{n/2\le k< n} \frac{n^4}{(k+1)^2(n-k)^2}$$
$$< \frac{2n^4}{(n/2)^2} \sum_{0\le k\le n/2} \frac{1}{(k+1)^2} = O(n^2).$$

Corr) $t_{mix} \le (2+o(1))\, n \log n$

Pf) Chebyshev w/ $t_0 = E(\tau) + 2\sqrt{\mathrm{Var}(\tau)}$
$P\{\tau > t_0\} \le 1/4$ ∎

Lower Bound (for random trans)

Prop 8.11 $t_{mix}(\varepsilon) \geq \frac{n-1}{2} \log\left(\frac{1-\varepsilon}{6} n\right)$

Pf Note: if $\rho \in S_n$ chosen unif @ random

$E[F(\rho)] = 1$

let $F(\sigma) := |\{i \in [n] \mid \sigma(i) = i\}|$

if σ is a product of t random transpositions

Then $F(\sigma) \geq$ (# elts not touched by any transposition)

$= R_{2t} =$ ~~#~~ coupons missing after $2t$ pulls in Coupon Collector.

let $\mu := n(1-1/n)^{2t} = E(R_{2t})$

$Var(R_{2t}) < \mu$

$A := \{\sigma \in S_n : F(\sigma) \geq \mu/2\}.$

$\pi(A) = 2/\mu$

$P^t(id, A^c) \leq P\{R_{2t} \leq \mu/2\} \overset{\text{Cheb}}{\leq} \frac{\mu}{(\mu/2)^2} = 4/\mu$

so $P^t(id, A) \geq 1 - 4/\mu$

so $\|P^t(id,\cdot) - \pi\|_{TV} \geq (1-4/\mu) - 2/\mu = 1 - 6/\mu.$

want: t small so that $1 - 6/\mu \geq \varepsilon$

or $n(1-1/n)^{2t} \geq \frac{6}{1-\varepsilon}$

$\Leftrightarrow (1-1/n)^{-2t} \leq \frac{n(1-\varepsilon)}{6}$

$\Leftrightarrow 2t \log \frac{n}{n-1} \leq \log \frac{n(1-\varepsilon)}{6}$

$\log(1+x) \leq x$

$\Leftarrow 2t \frac{1}{n-1} \leq \log \frac{n(1-\varepsilon)}{6}$ $\square$

8.3: Riffle Shuffles

More realistic!
3 methods:

① M binom$(n, \frac{1}{2})$ split deck into top M bottom $n-M$. pick unf at random one of the $\binom{n}{M}$ ways to interleave top & bot.

② top M, bot $n-M$ as above. at any time, if top pile has a cards
bot " " b "
then put bottom card of top pile down w/ prob $a/(b+a)$
& " " " " bot " " " $b/(b+a)$

③ to each card in deck, assign a unif random bit. Pull all 0's to top, keeping order otherwise

Dfn A <u>rising sequence</u> of $\sigma \in S_n$ is a maximal set of consecutive values that occur in the correct relative order ino

ex/

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 5 & 7 & 1 & 8 & 11 & 12 & 2 & 6 & 3 & 13 & 9 & 4 & 10 \end{pmatrix}$$
$$\begin{array}{ccccccccccccc} b & c & a & c & d & d & a & b & a & d & c & a & c \end{array}$$

Rising sequences:
a: 1 2 3 4
b: 5 6
c: 7 8 9 10
d: 11 12 13

Methods ① and ② both generate

$$Q(\sigma) = \begin{cases} n+1/2^n & \sigma = id \\ 1/2^n & \sigma \text{ has } =2 \text{ rising sequences} \\ 0 & \text{o.w.} \end{cases}$$

Claim: (3) generates $\hat{Q}(\sigma) = Q(\sigma^{-1})$.

Pf) The 0's and 1's form two rising sequences in σ^{-1}.

by Lemma 4.13, The distance from uniformity is The same for Q and $\hat{Q}$. So we will analyze (3).

Think of (3) differently:

~~match~~ pair each card w/ a bitstring, starting w/ the empty string.

at each t, ~~append~~ prepend a uniform random bit to each string, then perform a stable lexicographical sort.

let τ be the time until all bitstrings are different.

τ is a SST.

Prop 8.13. $t_{mix} \leq 2 \lg\left(\frac{4}{3} n\right)$ for n large

Pf) $P(\tau \leq t) = \prod_{k=0}^{n-1}\left(1 - \frac{k}{2^t}\right)$.

let $t = 2\log_2(n/c)$. $2^t = n^2/c^2$

$$\log \prod_{k=0}^{n-1}\left(1 - \frac{k}{2^t}\right) = -\sum_{k=0}^{n-1}\left(\frac{c^2 k}{n^2} + O\left(\frac{k}{n^2}\right)^2\right)$$

$$= \frac{c^2 n(n-1)}{2(n^2)} + O\left(\frac{n^3}{n^4}\right) = \frac{-c^2}{2} + O\left(\frac{1}{n}\right)$$

so $\lim_{n\to\infty} \frac{P(\tau < t)}{e^{-c^2/2}} = 1$

Thus, for any $c < \sqrt{2 \log 4/3} \approx 0.758\ldots$

gives a bound on t_{mix}. using $0.75 > 3/4$ gives what was wanted in the proposition.

i.e. $e^{-\log\frac{4}{3}} = 3/4$

Prop 8.14) fix $\varepsilon > 0$, $0 < \delta < 1$.

for large n

$$t_{mix}(\varepsilon) \geq (1-\delta) \lg n$$

Pf) at most 2^n states accessible in one step of chain.

Thus $\lg \Delta \leq n$ $\quad \Delta =$ max out-degree (7.1)

$$\lg(n!) = (1+o(1))\, n \lg n$$

So by 7.2

$$t_{mix}\,\varepsilon \geq \frac{\log(n!(1-\varepsilon))}{\log \Delta}$$

$$= \frac{\log(n!\,(1-\varepsilon))}{\log 2 \lg \Delta}$$

$$\geq \frac{\log((n!)(1-\varepsilon))}{\log 2 \cdot n}$$

$$= \frac{\lg n! + \lg 1-\varepsilon}{n}$$

$$\geq (1+o(1)) \lg n \quad \boxtimes$$