Chapter 6: Strong Stationary Times.

Example: Top to Random Shuffle.

Take top card & place it ~~ran~~ uniformly at random in the deck.
Let τ_{top} be the time one move after the first occasion when the original bottom card has moved to the top of deck.
Then we have

Prop 6.1
Let (X_t) be the random walk on S_n corresponding to top to random on n cards. Then the distribution of $X_{\tau_{top}}$ is uniform over S_n & τ_{top} is independent of $X_{\tau_{top}}$.

Pf: Claim: Given at time t, there are k cards under the original bottom card, each of the $k!$ possible orderings of these are equally likely.
(easy induction). This proves it.

Defn Given a sequence $(X_t)_{t=0}^{\infty}$ of Ω-valued RV's, a $\{0,1,\dots,\infty\}$-valued RV τ is a <u>stopping time</u> for (X_t) if for each $t \in \{0,1,\dots\}$, there is a set $B_t \subseteq \Omega^{t+1}$ s.t.

$$\{\tau = t\} = \{(X_0,\dots,X_t) \in B_t\}.$$

i.e. τ a stopping time <u>iff</u> $\mathbb{1}_{\{\tau = t\}}$ is a function of $X_0,\dots,X_t$ (what happened up to t).

~~ex.~~ Ex Hitting times, the first time MC hits a state is an example. $\tau_A = \min\{t \geq 0 : X_t \in A\}$ is a stopping time.
Exercise 6.1 $\Rightarrow$ $\tau + r$ is a stopping time where τ is a stopping time & r is a const. Hence $\tau_{top} = \tau_A + 1$ where A is ~~set~~ of arrangements w/ orig bottom on top.

<u>Defn</u> A <u>random mapping representation</u> of a transition matrix P on Ω is a function $f: \Omega \times \Lambda \to \Omega$ along w/ a Λ-valued RV, Z, satisfying

$$\Pr[f(x,Z) = y] = P(x,y)$$

<u>Note</u>: if $Z_1, Z_2, \ldots$ have same distribution as Z and X_0 has distribution μ, then the sequence (X_n) defined by

$$X_n = f(X_{n-1}, Z_n) \quad \text{for } n \geq 1$$

is a MC w/ transition matrix P & initial distrib μ.

<u>Prop 1.5</u>: Every transition matrix on a finite state sp. has a random mapping representation. (not unique)

<u>Defn</u> A random time τ is a <u>randomized stopping time</u> ~~if~~ ~~it is~~ for the Markov chain (X_n) if it is a stopping time for (Z_n) where (X_n), (Z_n) as in <u>Note</u> above.

Note that defn of stopping time does not require X_t's to be a MC or anything, so this makes sense.

<u>Clarifying example</u> (Random walk on hypercube) The lazy random walk (X_t) on $\{0,1\}^n$ can be constructed using the following random mapping rep: (j, B) is chosen u.a.r from $\{1,\ldots,n\} \times \{0,1\}$ and coordinate j is updated w/ bit B.

So let $Z_t = (j_t, B_t)$ be an i.i.d sequence used to update chain at step t.

Let $\tau_{refresh} := \min\{t \geq 0 : \{j_1, \ldots, j_t\} = \{1, \ldots, n\}\}$

(first time when all bits updated).

Then $X_{\tau_{refresh}}$ ~~is an exact~~ has uniform (stationary) distribution.

$\tau_{refresh}$ is a randomized stopping time.

Defn Let (X_t) be an irreducible MC w/ stationary distribution π. A stationary time τ for (X_t) is a randomized stopping time, possibly depending on starting position x, s.t. the distribution of X_τ is π:

i.e. $P_x[X_\tau = y] = \pi(y)$.

Ex 6.6 Let (X_t) be the random walk on n-cycle. (uniform stationary). Define τ by tossing a coin w/ prob heads $= \frac{1}{n}$. If heads, let $\tau = 0$.
If tails let τ = first time each state visited at least once. Exercise 6.9 shows given tails, distribution is uniform on verts other than start. So X_τ has uniform distribution. So τ is a stationary time.

However, $\tau = 0 \Rightarrow X_\tau = X_0$ (start state), so τ & X_τ not indep. We ~~want~~ need this property to be able to bound t_{mix}.

Defn: A strong stationary time for MC (X_t) w/ stationary distrib π, is a randomized stopping time τ (possibly depending on start state x, s.t.

$$P_x\{\tau = t, X_\tau = y\} = P_x\{\tau = t\} \cdot \pi(y). \quad (\text{indep.})$$

τ_{top} & $\tau_{refresh}$ satisfy this.

The main tool is

Prop 6.10 If τ is a strong stationary time, then

$$\left(\max_x \|P^t(x,\cdot) - \pi\|_{TV} = \right) d(t) \le \max_{x \in \Omega} P_x\{\tau > t\}.$$

Proof is via 3 lemmas.

Lemma 6.9 Let (X_t) be an irreducible MC w/ stationary distrib π. If τ is a strong stationary time for (X_t), then for all $t \geq 0$

$$P_x\{\tau \leq t, X_t = y\} = P_x\{\tau \leq t\}\pi(y)$$

Pf (idea) Let $Z_1, Z_2 \ldots$ be the i.i.d random mapping seq. For any $s \leq t$

$$P_x\{\tau = s, X_t = y\} = \sum_{z \in \Omega} \underbrace{P_x\{X_t = y \mid \tau = s, X_s = z\}}_{P^{t-s}(z,y)} \underbrace{P_x\{\tau = s, X_s = z\}}_{\pi(z) P_x\{\tau = s\}}$$

since the $\tau = s$ depends on $Z_1, \ldots, Z_s$ & $X_t = y$ depends on $Z_{s+1}, \ldots, Z_t$.

but $\pi = \pi P^{t-s} \Rightarrow$

$$P_x\{\tau = s, X_t = y\} = \pi(y) P_x\{\tau = s\}.$$

So summing over $s \leq t$ gives it.

Lemma 6.11 If τ is a strong stationary time, then

$$S_x(t) \leq P_x\{\tau > t\}.$$

where $S_x(t) = \max_{y \in \Omega}\left[1 - \frac{P^t(x,y)}{\pi(y)}\right]$ (called separation distance).

Pf: Fix $x \in \Omega$. $\forall y$

$$1 - \frac{P^t(x,y)}{\pi(y)} = 1 - \frac{P_x\{X_t = y\}}{\pi(y)}$$

$$\leq 1 - \frac{P_x\{X_t = y, \tau \leq t\}}{\pi(y)}$$

$$= 1 - \frac{P_x\{\tau \leq t\}\cancel{\pi(y)}}{\cancel{\pi(y)}} = P_x\{\tau > t\}.$$

Lemma 6.13 $\|P^t(x,\cdot)-\pi\|_{TV} \le s_x(t)$.

Pf: $$\|P^t(x,\cdot)-\pi\|_{TV} = \sum_{\substack{y\in\Omega \\ P^t(x,y)<\pi(y)}} [\pi(y)-P^t(x,y)]$$ Rmk 4.3

$$= \sum_{\substack{y\in\Omega \\ P^t(x,y)<\pi(y)}} \pi(y)\left(1-\frac{P^t(x,y)}{\pi(y)}\right)$$

$$\le \max_y \left(1-\frac{P^t(x,y)}{\pi(y)}\right) = s_x(t).$$

Applications

Random Walk on Hypercube.

Recall coupon collector. τ is total # of coupons when we have all types.

Then Prop 2.4 $P[\tau > \lceil n\log n + cn\rceil] \le e^{-c}$.

Then $\tau_{refresh}$ is such a variable where "coupons" are bits ~~#~~ updating.

$t_{mix}(\varepsilon) = \min\{t : d(t) \le \varepsilon\}$.

we get by prop 6.10 $$d(n\log n + cn) \le \max_{x\in\Omega} P_x\{\tau_{refresh} > n\log n + cn\}$$
$$\le e^{-c}.$$

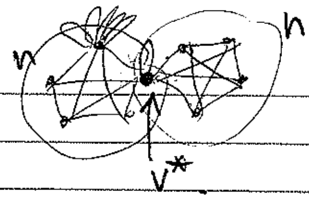
let $\varepsilon = e^{-c}$ to get $t_{mix}(\varepsilon) \le n\log n + \log(\frac{1}{\varepsilon})n$.

Top to Random shuffle

Distribution of τ_{top} is same as coupon collector. again.

So we get $d(n\log n + cn) \le e^{-c}$ $\forall n$

so $t_{min}(\varepsilon) \le n\log n + \log(\frac{1}{\varepsilon})n$.



Glued complete graphs

It is regular of deg $2n-1$. (uniform stationary)

If τ is one step after v^* visited, ~~then~~ for first time, then τ is strong stationary time.

If walk is not at v^*, then prob of moving to v^* in a step is $\frac{1}{2n-1}$. ~~(geometric~~

So τ_{v^*} is geometrically distrib & $E[\tau_{v^*}] = 2n-1$.

So $E[\tau] = 2n$. So Markov $\Rightarrow P_x\{\tau \geq t\} \leq \frac{2n}{t}$

Take $t = 8n$ to get

$$d(8n) \leq \max_x P_x\{\tau > 8n\} \leq P_x\{\tau \geq 8n\}$$
$$\leq \frac{2n}{8n} = 1/4.$$

So $t_{mix} = t_{mix}(\tfrac{1}{4}) \leq 8n$.