4\28\10

Path Coupling

We have a strongly connected digraph $D = (\Omega, A)$

$\ell : A \to \mathbb{R}^+ ; \quad \ell \geq 1$

$\rho(X,Y)$ = shortest distance $X \to Y$ using ℓ.

[Often $\rho(x,y) = \#(i : x_i \neq y_i) = h(x,y)$

$$\rho(x,y) \geq 1_{\{x \neq y\}}$$

Coupling: $\alpha > 0$

$$E_{X,Y}\left(\rho(X', Y')\right) \leq e^{-\alpha} \rho(X, Y)$$

$(X,Y) \in A$: $(X,Y) \longrightarrow (X', Y')$

a coupling of the chain.

$\Downarrow$ to be proved

$\forall$ pairs X, Y

$$E_{X,Y}\left(\rho(X_t, Y_t)\right) \leq e^{-\alpha t} \operatorname{diam}(\Omega)$$

So

$$\|P^t(x,\cdot) - P^t(y,\cdot)\|_{TV}$$

$$\leq P_{x,y}\{X_t \neq Y_t\}$$

$$= P_{x,y}\{\rho(X_t, Y_t) \geq 1\}$$

$$\leq E_{x,y}\,\rho(X_t, Y_t)$$

$$\leq e^{-\alpha t}\,\mathrm{diam}(\Omega)$$

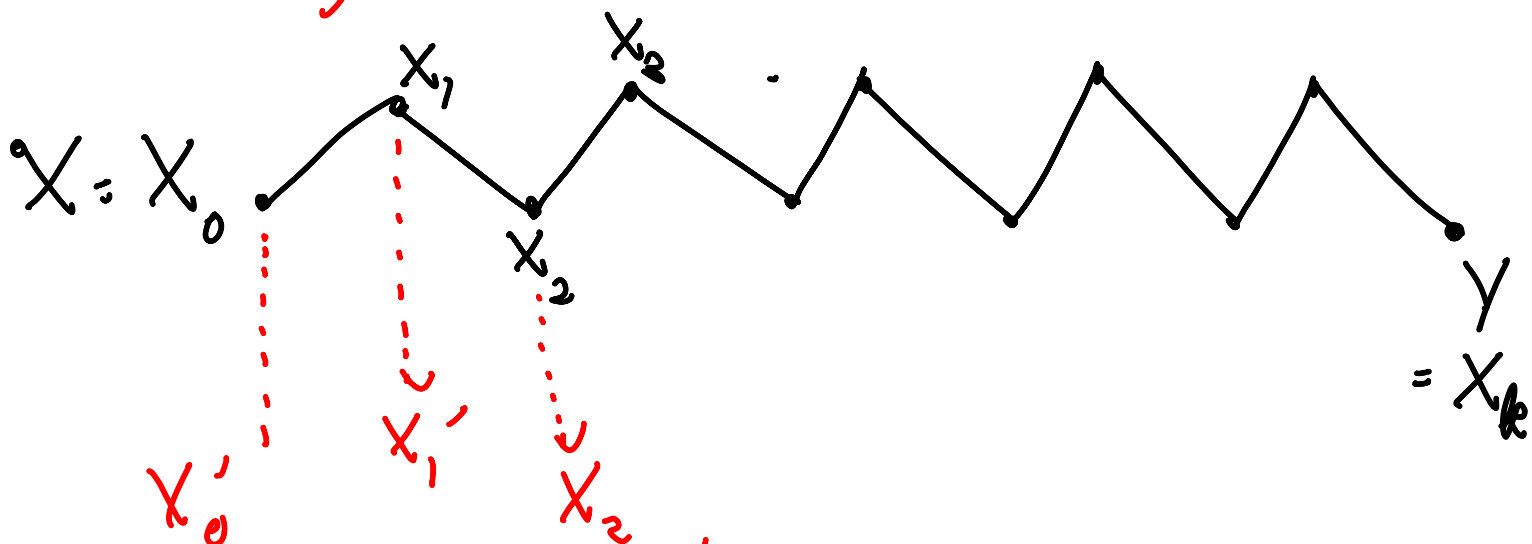
$$t_{mix}(\epsilon) \leq \left\lceil \frac{1}{\alpha}\left(\log(\mathrm{diam}\,\Omega) + \log\frac{1}{\epsilon}\right\rceil$$

Proof.

We show

$$E_{x,y}\left(\rho(x',y')\right) \leq e^{-\alpha}\rho(x,y)$$

for all x, y.



$E(x_0', x_1')$ $\leq e^{-\alpha}\rho(x_0, x_1)$ $\quad$ $E(x_1', x_2')$ $\leq e^{-\alpha}\rho(x_1, x_2)$ $\quad$ - - - - -

$$E(\rho(X', Y'))$$

$$\leq E\left(\rho(X'_0, X'_1) + \rho(X'_1, X'_2) + \cdots\right)$$

$$\leq e^{-\alpha}\left[\rho(X_0, X_1) + \cdots\right)$$

$$= e^{-\alpha}\rho(X, Y)$$

Colourings of Graph

Suppose $q > 2\Delta$

$\Omega = \{$ proper q-colorings of $G \}$

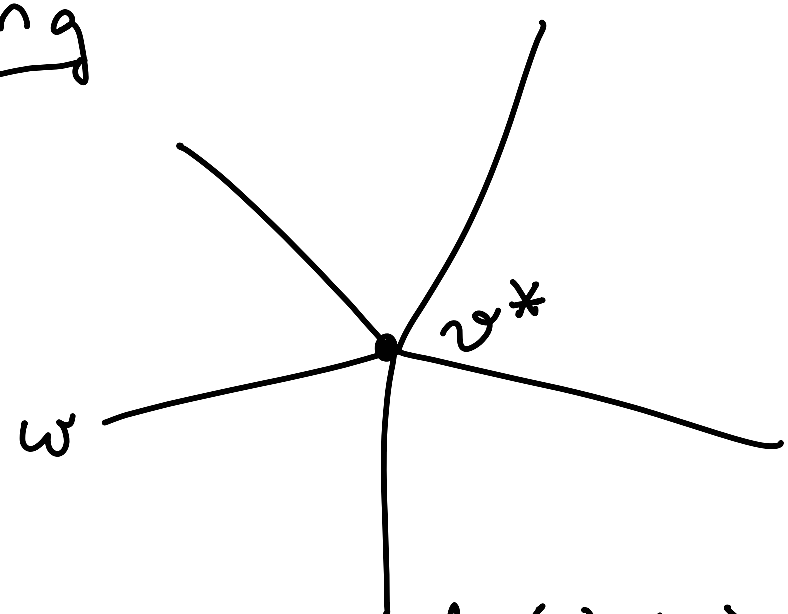
Chain: Glauber Dynamics

(i) Choose v at random

(ii) Randomly re-color with available color.

$D = \{ \Omega, \{ (X,Y) : h(X,Y) = 1 \} \}$

Coupling



$X(v^*) \neq Y(v^*)$

$X(w) = Y(w)$,

$w \neq v^*$

$$\rho(X, Y) = h(X, Y)$$

X, Y choose same random vertex v

(i) $v = v^*$; re-color v with same color & $X' = Y'$

(ii) $v \notin v^* \cup N(v^*)$: choose same color $h(X', Y') = 1$

(iii) $v = w \in N(v^*)$.

$A_w(X)$ = colors available to w in X

$A_w(Y)$ = colors available to w in Y

$$A_w(X) \setminus A_w(Y) \subseteq \{Y(v^*)\}$$

$$A_w(Y) \setminus A_w(X) \subseteq \{X(v^*)\}$$

Assume $|A_w(X)| \leq |A_w(Y)|$

(i) Choose $c = Y'(w)$ uniformly from $A_w(Y)$

(ii) If $c \neq X(v^*)$ then $X'(w) \leftarrow c$

else if $c = X(v^*)$

(a) $|A_w(X)| = |A_w(Y)|$: $X'(w) = Y(v^*)$

[$A_w(X) \oplus A_w(Y) = \{X(v^*), Y(v^*)\}$

(b) Choose randomly from $A_w(X)$.

Make sure X, Y are properly updated.

(1) Clear that Y is properly updated.

$\underline{X:}$

(a) Suppose $a = |A_w(X)|$.

$$\begin{array}{cccccc} c_1 & c_2 & c_3 & \cdots & & c_a = Y(v^*) \\ \uparrow & \uparrow & & & & \uparrow \\ Pr = 1/a & 1/a & & & & \frac{1}{a} \text{ we choose } X(v^*) \end{array}$$

(b) $\text{Prob}[\text{color}] = \dfrac{1}{a+1} + \underbrace{\dfrac{1}{a+1}}_{c = X(v^*)} \cdot \dfrac{1}{a} = 1/a$

$$\frac{A_w(Y)}{\overline{A_w(X)}}$$

$$E\left(h(X', Y') \mid X, Y\right)$$

$|A_w(x)| \geq q - \Delta$

$$\leq 1 - \frac{1}{n} + \frac{\Delta}{n} \cdot \frac{1}{q-\Delta}$$

$$\leq 1 - \frac{1}{n}\left(1 - \frac{\Delta}{q-\Delta}\right)$$

$$\leq 1 - \frac{1}{n} \cdot \frac{1}{\Delta+1}$$

$$\leq e^{-\frac{1}{\underbrace{n(\Delta+1)}_{\alpha}}} \implies \text{Mixing Time} = O(n \log n)$$

Diam (Ω) ?

(1) $\Omega' = \{\text{colourings}\} \supseteq \Omega$

$|\Omega'| = q^{|V|}$

Non-proper colorings have $\pi = 0$.

$\mathrm{diam}(\Omega) \leq n$