4/21/10

Lemma 13.12

$\psi : \Omega \to \mathbb{R}^+$ such $\psi_1 \geq \psi_2 \geq \cdots \geq \psi_m > 0 = \psi_{m+1} = \cdots$

$\uparrow$

$\{1, 2, \cdots, n\}$

$\pi_1 + \pi_2 + \cdots + \pi_m \leq \frac{1}{2}$

$$\langle \psi, \mathbb{1} \rangle_\pi \leq \frac{1}{\Phi_*} \sum_{x<y} |\psi(x) - \psi(y)| \, Q(x,y)$$

Now suppose $Pf_2 = \lambda_2 f_2$ and

$$\pi\{f_2 > 0\} \leq \frac{1}{2}$$ [else take $-f_2$]

$$f = \max\{f_2, 0\}$$

(i) $f(x) = 0$ & $Pf(x) \geq 0$

$$(I-P) f(x) \leq \gamma f(x)$$

(ii) $f(x) = f_2(x) > 0$ &

$(I-P) f_2(x) = \gamma f_2(x)$

$$\Downarrow$$

$$\langle (I-P) f, f \rangle_\pi \leq \gamma \langle f, f \rangle_\pi$$

$$\boxed{R \leq \gamma}$$

Apply (13.1) to $\psi = f^2$.

$$\langle f, f \rangle_\pi^2 \leq \Phi_*^{-2} \left(\sum_{x<y} \left(f^2(x) - f^2(y) \right) Q(x,y) \right)^2$$

$\downarrow$ Cauchy-Schwartz

$$\leq \Phi_*^{-2} \left(\sum_{x<y} \left(f(x) - f(y) \right)^2 Q(x,y) \right) \times$$

$$\left(\sum_{x<y} \underbrace{\left(f(x) + f(y) \right)^2}_{2\left[f(x)^2 + f(y)^2\right] - \left[f(x) - f(y)\right]^2} Q(x,y) \right)$$

$$= \Phi_*^{-2} \langle (I-P) f, f \rangle_\pi \left(2\langle f,f\rangle_\pi - \langle (I-P) f, f\rangle_\pi \right)$$

Since $R = \dfrac{\langle (I-P) f, f \rangle_\pi}{\langle f, f\rangle_\pi} \leq \gamma$

then dividing by $\langle f, f \rangle_\pi^2$ we get

$$1 \leq \Phi_*^{-2} R(2-R)$$

$$1 - \Phi_*^2 \geq (1-R)^2 \geq (1-\gamma)^2$$

So $\left(1 - \tfrac{1}{2}\Phi_*^2\right) \geq (1-\gamma)^2.$ $\square$

Example

Random walk on an r-regular expander graph.

Suppose G satisfies $|S:\bar{S}| \geq \alpha |S|$ for $|S| \leq \frac{1}{2}|V| = \frac{1}{2}n$.

\# edges leaving $\geq \alpha |S|$.

$$\Phi(S) = \frac{\displaystyle\sum_{\substack{x \in S \\ y \in \bar{S}}} \frac{1}{n} \cdot \mathbb{1}_{\{(x,y) \in E\}} \cdot \frac{1}{r}}{\frac{1}{n} \cdot |S|} = \frac{|S:\bar{S}|}{r|S|} \geq \frac{\alpha}{r}$$

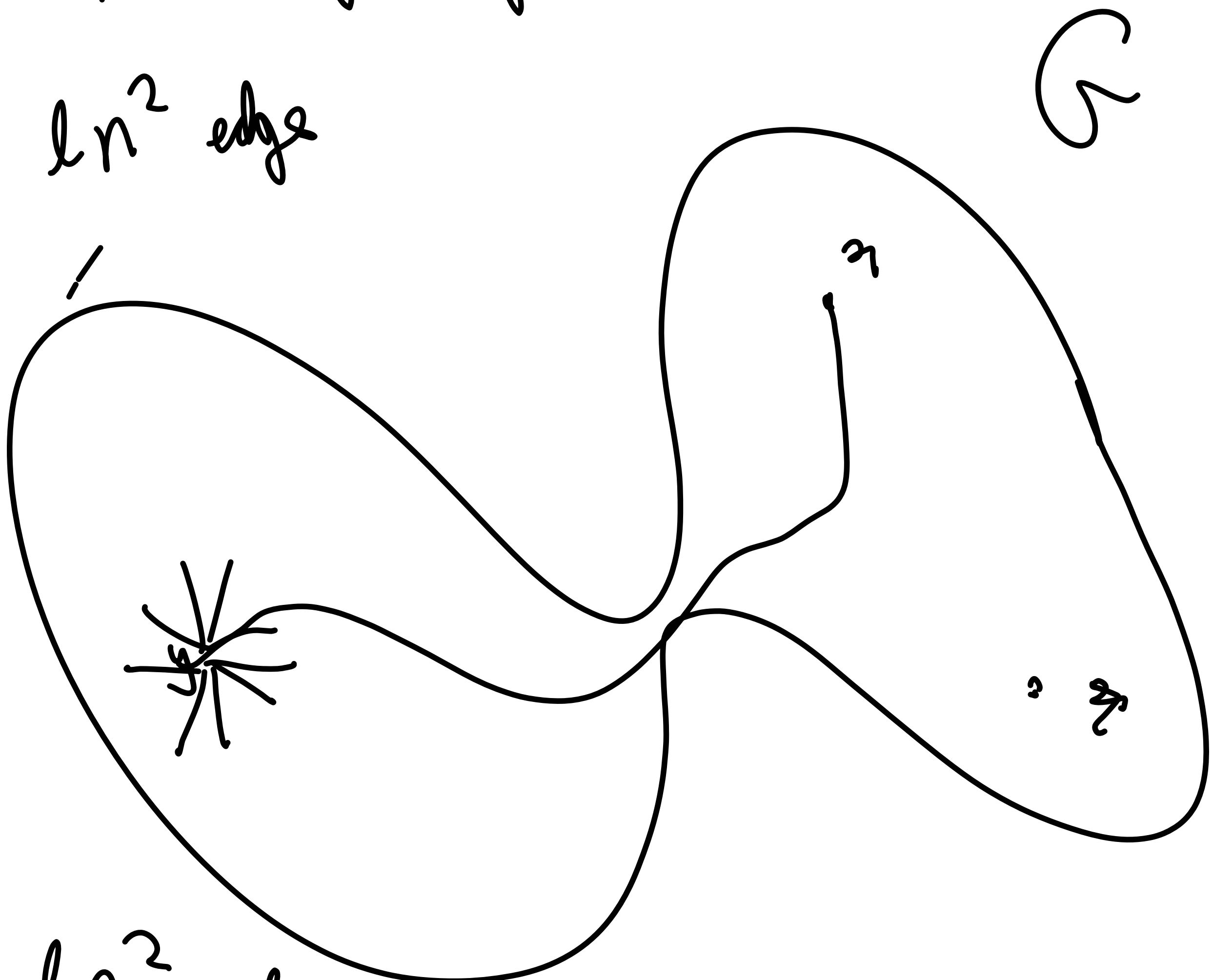
$$\lambda_2 \leq 1 - \frac{\alpha^2}{2r^2}$$

<u>Lazy Walk</u>

$$\Rightarrow d(t) \leq n\left(1 - \frac{\alpha^2}{2r^2}\right)^t .$$

n^2 paths of length l

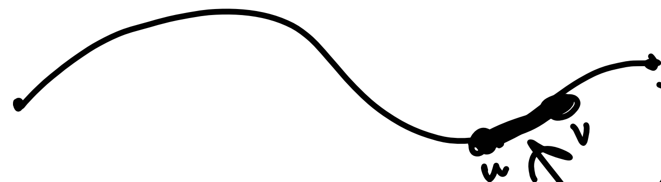
ln^2 edge



$\frac{ln^2}{|E|} \approx \frac{ln}{c}$ $\quad |E| \approx cn$

Canonical Paths

Suppose we have a collection of paths

$\{\gamma_{xy} :$ x  $y\}$

$P(u,v) > 0.$

$x, y \in \Omega.$

$$\rho = \max_{\underbrace{P(u,v)>0}_{e}} \left\{ \frac{1}{\pi(u) P(u,v)} \sum_{\gamma_{xy} \ni e} \pi(x)\pi(y) |\gamma_{x,y}| \right\}$$

length of $\gamma_{x,y}$

$$\therefore \quad \frac{\mathcal{E}_\pi(f,f)}{\mathrm{Var}_\pi(f)} \geq \frac{1}{\rho}$$

$$2\,\mathrm{Var}_\pi(f) = \sum_{x,y} \pi(x)\pi(y)\big(f(x)-f(y)\big)^2$$

$$= \sum_{x,y} \pi(x)\pi(y)\Bigg(\sum_{(u,v)\in\gamma_{xy}} \big(f(u)-f(v)\big)\Bigg)^2$$

Cauchy Schwartz

$$\leq \sum_{x,y} \pi(x)\pi(y)\,|\gamma_{x,y}| \sum_{u,v\in\gamma_{x,y}} \big(f(u)-f(v)\big)^2$$

$$= \sum_{u,v} \sum_{\substack{x,y \\ \gamma_{x,y}\ni(u,v)}} \pi(x)\pi(y)\,|\gamma_{x,y}|\,\big(f(u)-f(v)\big)^2$$

$$= \sum_{u,v} \sum_{\substack{x,y \\ \gamma_{x,y} \ni (u,v)}} \pi(x)\pi(y)\,|\gamma_{x,y}|\,\big(f(u)-f(v)\big)^2$$

$$= \sum_{u,v} \big(f(u)-f(v)\big)^2 \sum_{\substack{x,y \\ \gamma_{x,y} \ni (u,v)}} \pi(x)\,\pi(y)\,|\gamma_{x,y}|$$

$$\leq \sum_{u,v} \big(f(u)-f(v)\big)^2 \pi(u) P(u,v)\, \rho$$

$$= 2\rho\, \mathcal{E}_\pi(f,f).$$

Matchings

$G = (V, E)$ and $\mathcal{M} = \{\text{matchings of } G\}$

set of disjoint edges of G.

Problem: Generate a (near) random member of $\mathcal{M}$.

Markov Chain

$X_t \in \mathcal{M}$;

For matchings M_1, M_2

$P[M_1, M_2] = P[M_2, M_1] = 0$ or $\frac{1}{2|E|}$

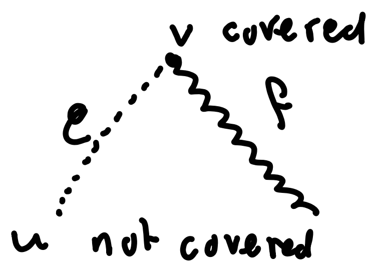
Uniform steady state.

(0) $X_{t+1} = X_t$; Prob $\frac{1}{2}$; lazy chain

(1) Choose $e = (u,v) \in E$ at random

(a) $e \in X_t$; $X_{t+1} = X_t - e$ $\downarrow$

(b) u, v both not covered by X_t ; $X_{t+1} = X_t + e$ $\uparrow$

(c) 

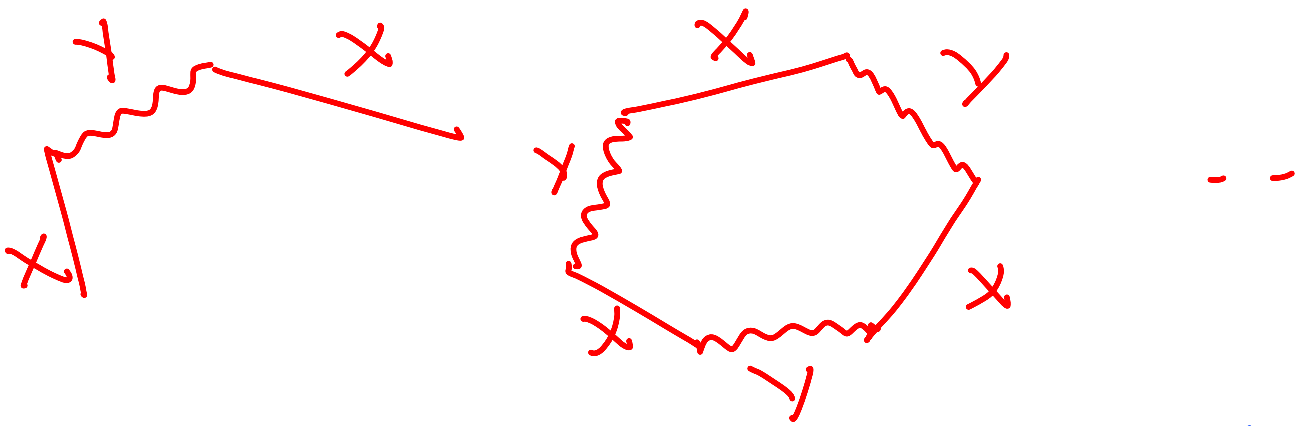
; $X_{t+1} = X_t - f + e$ $\leftrightarrow$

(d) u, v both covered ; $X_{t+1} = X_t$

We define canonical paths $\gamma_{x,y}$ on

$$\Gamma = \left(\mathcal{M},\ A = \{(x,y) : x \to y \text{ is a possible transition}\}\right)$$

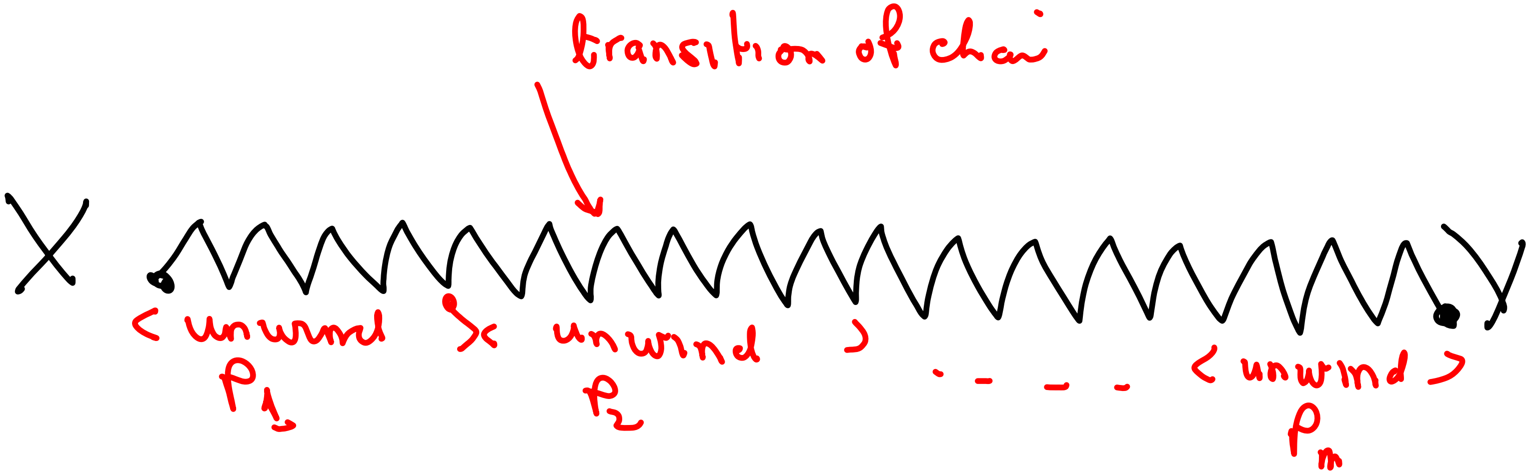
$$x \oplus y = (x/y) \vee (y/x)$$



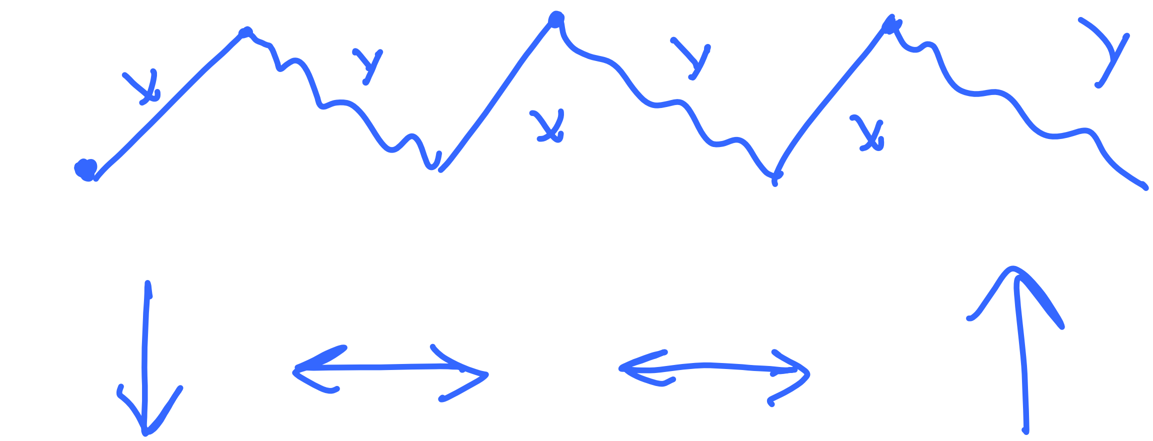
Assume some ordering of the set of paths and cycles of Γ.

$$X \oplus Y = P_1 + P_2 + \cdots + P_n$$

in the ordering of paths.

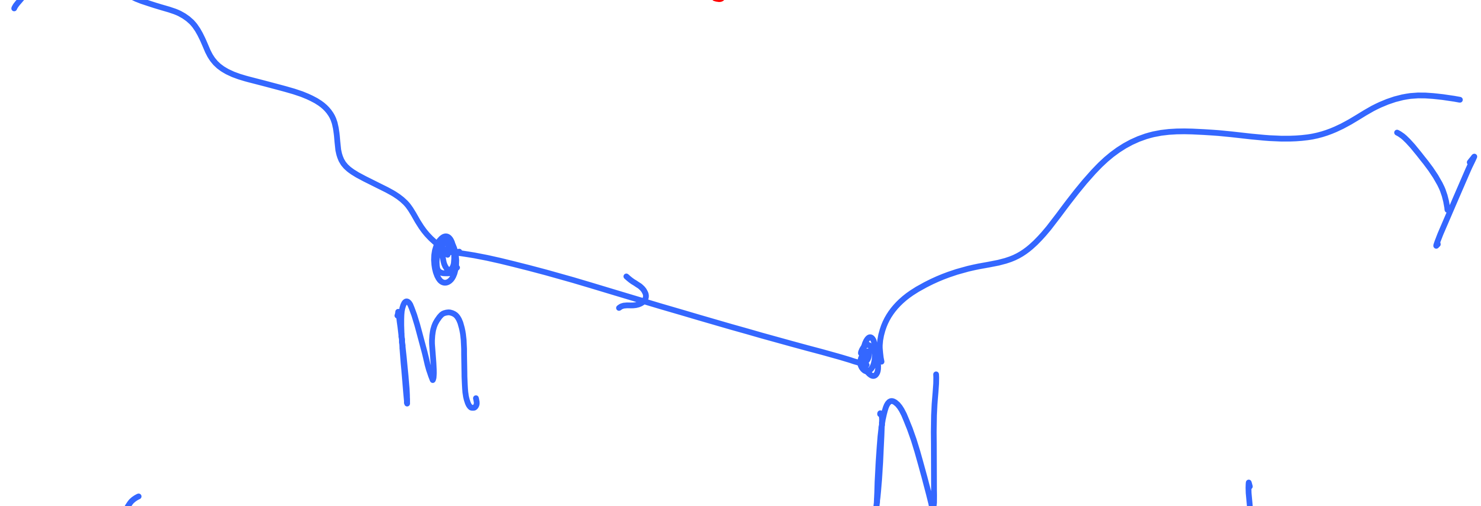


"unwind" — delete the X's and the Y's



Fix (M,N)

Injection $\{(X,Y)\} \to M$



$$\#(X,Y) \text{ s.t. } (M,N) \in \gamma_{X,Y}) \leq |M|$$