4\19\10

Dirichlet Form

Given P, π define

$$\mathcal{E}(f,h) = \langle (I-P)f, h \rangle_\pi$$

L 13.11

$$\mathcal{E}(f,f) = \frac{1}{2} \sum_{x,y \in \Omega} \big(f(x) - f(y)\big)^2 Q(x,y)$$

if P is reversible

Proof

$$\text{RHS} = \frac{1}{2}\sum_{x,y} f(x)^2 Q(x,y) - \sum_{x,y} f(x) f(y)\, Q(x,y)$$

$$+ \frac{1}{2}\sum_{x,y} f(y)^2 Q(x,y)$$

$$\sum_{x} f(x)^2 \pi(x) \sum_{y} P(x,y) = \sum_{x} f(x)^2 \pi(x)$$

$$\sum_{y} f(y)^2 \sum_{x} \pi(x) P(x,y) = \sum_{y} f(y)^2 \pi(y) \underbrace{\sum_{x} P(y,x)}_{1}$$

$$\text{RHS} = \sum_x f(x)^2 \pi(x) - \sum_x f(x) \left(\sum_y f(y) P(x,y) \right) \pi(x)$$

$$= \langle f, f \rangle_\pi - \langle f, Pf \rangle_\pi$$

$$= \mathcal{E}(f,f)$$

Rayleigh Quotient

L13.12 $\quad \gamma = 1 - \lambda_2$

$$\gamma = \min_{\substack{f \in \mathbb{R}^\Omega \\ f \neq 0 \\ \langle f, \mathbb{1} \rangle_\pi = 0}} \frac{\mathcal{E}(f,f)}{\langle f, f \rangle_\pi} = \min_{f \neq 0} \frac{\mathcal{E}\big(\overbrace{f - E_\pi(f)}^{g}, f - E_\pi(f)\big)_\pi}{\mathrm{Var}_\pi(f)}$$

$$\mathrm{Var}_\pi(f) = \mathrm{Var}_\pi(g)$$

$$E_\pi(f) = 0$$

Proof

right eigenvectors of P

Write

$$f = \sum_{j=1}^{n} \underbrace{\langle f, f_j \rangle_\pi}_{a_j} f_j$$

Can assume that $\langle f, f \rangle_\pi = 1$.

$f_1 = 1$

$\langle f, 1 \rangle_\pi = 0$

$$1 = \langle f, f \rangle_\pi = \sum_{j=1}^{n} \langle f, f_j \rangle^2$$

$$= \sum_{j=2}^{n} a_j^2$$

$$\langle (I-P) f, f \rangle_{\pi} = \sum_{j=2}^{n} a_j^2 (1-\lambda_j)$$

$$\geq 1-\lambda_2$$

$$\frac{\langle (I-P) f, f \rangle_{\pi}}{Var_{\pi}(f)} \geq 1-\lambda_2 .$$

Rayleigh Quotient $\geq \gamma$

$f = f_2$: $\langle (I-P) f_2, f_2 \rangle = 1-\lambda_2$

scaled to $\langle f_2, f_2 \rangle = 1$

$$\Phi_* = \min_{\pi(S) \leq \frac{1}{2}} \frac{Q(S, \bar{S})}{\pi(S)} \leftarrow \Phi(S).$$

conductance

Thm 13.14

Cheeger Inequality

$$\frac{1}{2}\Phi_*^2 \leq \gamma \leq 2\Phi_*$$

$1 - \lambda_2$ ← only spectral gap if $\lambda_2 = \lambda_{max}$

$\lambda_2 = \lambda_{max}$ for a lazy chain

$$P = \frac{\hat{P} + I}{2}$$

$\hat{P}$ is stochastic

$$\lambda_j = \frac{1 + \hat{\lambda}_j}{2} \geq 0$$

Upper Bound

$$\gamma = \min_{\substack{f \neq 0 \\ \langle f, 1 \rangle_\pi = 0}} \frac{\sum_{x,y} \pi(x) P(x,y) \left[f(x) - f(y) \right]^2}{\sum_{x,y} \pi(x) \pi(y) \left(f(x) - f(y) \right)^2}$$

Variance

Variance $= E((X - \bar{X})^2)$

If Y, Z are independent copies of X

$$E((Y - Z)^2) = E(Y^2) + E(Z^2) - 2E(Y)E(Z)$$

$$= 2 \text{ Variance}(X).$$

If $S \subseteq \Omega$ let

$$f_S(x) = \begin{cases} -\pi(S^c) & x \in S \\ \pi(S) & x \in S^c \end{cases}$$

$$\boxed{\sum_x f_S(x)\,\pi(x) = 0}$$

$$\leq \Phi(S).$$

$$\gamma \leq \frac{\displaystyle\sum_{\substack{x\in S\\ y\in S^c}} \pi(x)P(x,y) + \sum_{\substack{x\in S^c\\ y\in S}} \pi(x)P(x,y)}{\displaystyle\sum_{\substack{x\in S\\ y\in S^c}} \pi(x)\pi(y) + \sum_{\substack{x\in S^c\\ y\in S}} \pi(x)\pi(y)}$$

$$Q(S,S^c) + Q(S^c,S) = 2Q(S,S^c)$$

$$2\pi(S)\pi(S^c)$$

Lower Bound

L13.17

$\psi : \underbrace{\Omega}_{\{1,2,\dots,n\}} \to \mathbb{R}$

$\psi_1 \geq \psi_2 \geq \dots \geq \psi_m > 0 \geq \psi_{m+1}$

$\dots \geq \psi_n \geq 0$

$\pi_1 + \pi_2 + \dots + \pi_m \leq \frac{1}{2}$

Then

$$\langle \psi, \mathbb{1} \rangle_\pi \leq \frac{1}{\Phi_*} \sum_{x<y} |\psi(x) - \psi(y)| \, Q(x,y)$$

as a number

$$S_t = \{x : \psi(x) > t\}$$

$$\Phi_* \le \frac{\sum_{x,y} Q(x,y)\, \mathbb{1}_{\{\psi(x) > t \ge \psi(y)\}}}{\pi(\psi > t)}$$

$$\Rightarrow$$

$$\underbrace{\pi(\psi > t)}_{\Downarrow\ \langle \psi, \mathbb{1} \rangle_\pi} \le \frac{1}{\Phi_*} \sum_{x<y} Q(x,y)\, \underbrace{\mathbb{1}_{\{\psi(x) > t > \psi(y)\}}}_{\psi(x) - \psi(y)}$$

$\Downarrow$ integrate over t

$$\int_{t=0}^{\infty} \pi\{\psi > t\}\, dt = \int_{t=0}^{\infty} \sum_{x} \pi(x) \mathbb{1}_{\{\psi(x) > t\}}\, dt$$

$$= \sum_{x} \int_{t=0}^{\infty} \mathbb{1}_{\{\psi(x) > t\}}\, \pi(x)\, dt$$

$$= \sum_{x} \pi(x) \psi(x)$$

$$= \langle \psi, \mathbb{1} \rangle_{\pi}$$

$$\int_{t=0}^{\infty} \mathbb{1}_{\{\psi(x) > t > \psi(y)\}}\, dt$$

$$= \psi(x) - \psi(y)$$