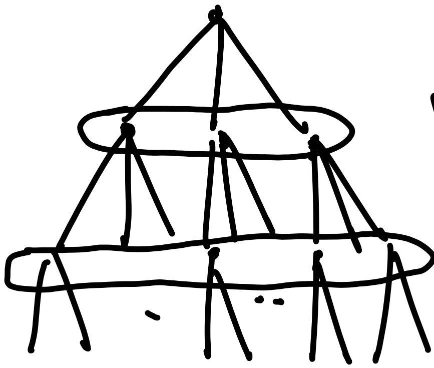
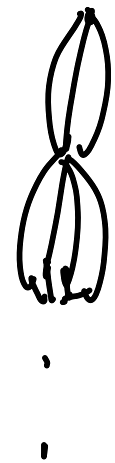
3/31/10

Ex 9.8

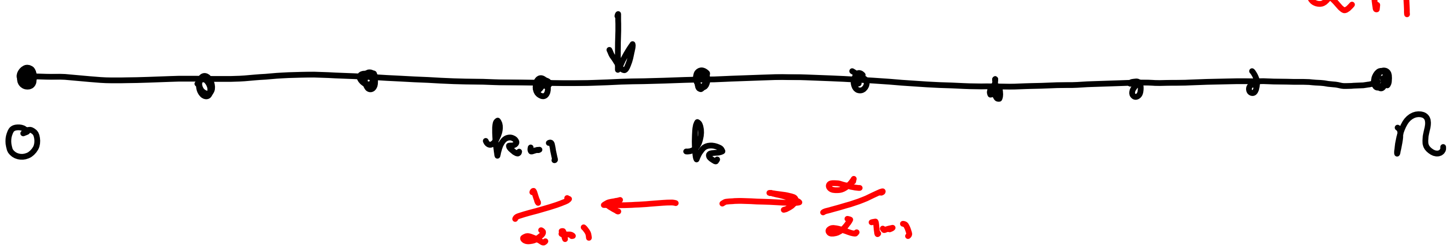


degree and resistance only depend on depth

$\xRightarrow{\text{Glue}}$

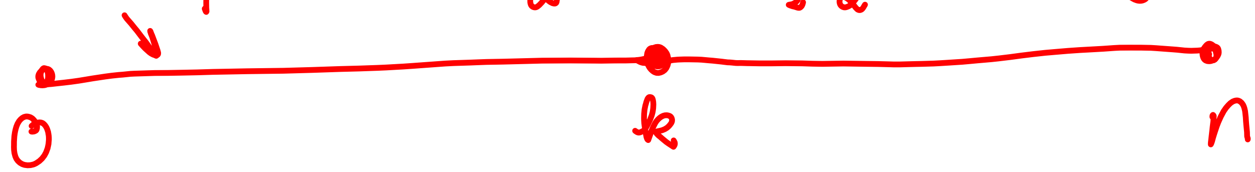


$\alpha > 0$ $\quad c(k-1,k) = \alpha^k$ $\quad P(k,k+1) = \frac{\alpha}{\alpha+1}$



Compute $P_k(T_n < T_0) = \dfrac{W(n)-W(k)}{W(n)-W(0)}$ (9.14)

$r_1 = 1 + \frac{1}{\alpha} + \cdots + \frac{1}{\alpha^k}$ $\quad r_2 = \frac{1}{\alpha^{k+1}} + \cdots + \frac{1}{\alpha^n}$



$$\frac{r_2}{r_1 + r_2}$$

$W(n) = 1$

$W(0) = 0$ $\ni$ $\|I\| = \frac{1}{r_1+r_2}$

$W(k) = \frac{r_1}{r_1+r_2}$

Energy

$$\mathcal{E}(\theta) = \sum_e \theta(e)^2 r(e)$$

flow

Theorem

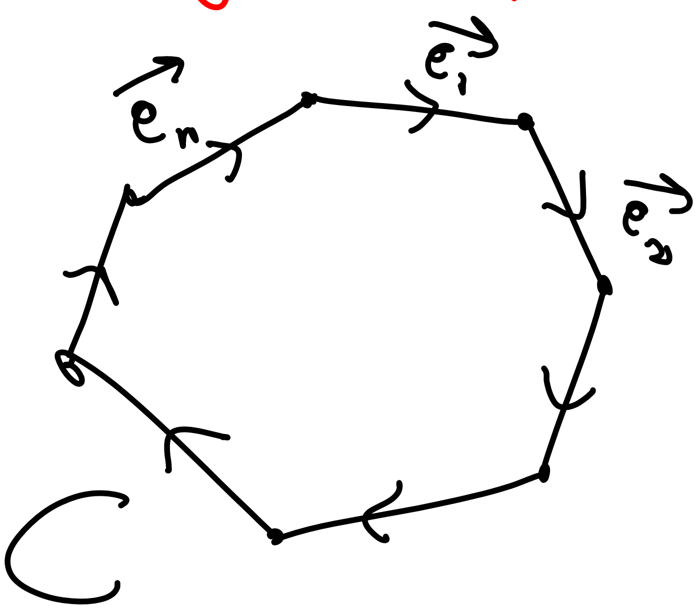
$$R(a \leftrightarrow z) = \inf_{\theta} \{ \mathcal{E}(\theta) : \theta \text{ is a unit flow from } a \text{ to } z. \}$$

The minimising flow is the unit current

Proof

(i) $\exists\, \theta$ minimising $\mathcal{E}(\theta)$

From Prop. 9.4 it is enough to show that unit flow of minimum energy satisfies the cycle law.



$\gamma(\vec{e_i}) = 1, \quad 1 \le i \le n$

$\gamma(e) = 0, \quad e \notin C$

Consider $\underbrace{\theta + \epsilon\gamma}_{\text{flow}}$

$$0 \leq \mathcal{E}(\theta + \epsilon\gamma) - \mathcal{E}(\theta) = \sum_{i=1}^{n} \left[\left(\theta(\vec{e}_i) + \epsilon\right)^2 - \theta(\vec{e}_i)^2 \right] r(\vec{e}_i)$$

$$= 2\epsilon \underbrace{\sum_{i=1}^{n} r(\vec{e}_i)\,\theta(\vec{e}_i)} + O(\epsilon^2)$$

If this is positive, we can get a contradiction by making ϵ small and negative.

Similarly for $\sum < 0$.

Finally,

$$\sum_e r(e)\, I(e)^2 = \frac{1}{2} \sum_x \sum_y r(x,y) \left(\frac{W(x)-W(y)}{r(x,y)} \right)^2$$

$$= \frac{1}{2} \sum_x \sum_y c(x,y) \left(W(x) - W(y) \right)^2$$

$$= \frac{1}{2} \sum_x \sum_y \left(W(x) - W(y) \right) I(\overrightarrow{xy})$$

$$= \sum_x W(x) \underbrace{\sum_y I(\overrightarrow{xy})}_{0 \text{ for } x \neq a, z} \quad \text{since } I(\overrightarrow{xy}) = -I(\overrightarrow{yx})$$

$$= \underbrace{\|I\|}_{=1} \left(W(a) - W(z) \right)$$

$$= \frac{W(a) - W(z)}{\|I\|} = R(a \leftrightarrow z) \qquad \square$$

__Thm 12__ [Rayleighs Monotonicity Law]

If $r(e) \leq r'(e)$ then $R(a \leftrightarrow z) \leq R'(a \leftrightarrow z)$

__Proof__

$$\inf_{\theta} \sum_{e} r(e)\, \theta^2(e) \leq \inf_{\theta} \sum_{e} r'(\theta)\, \theta(e)^2$$

__Cor. 9.13__

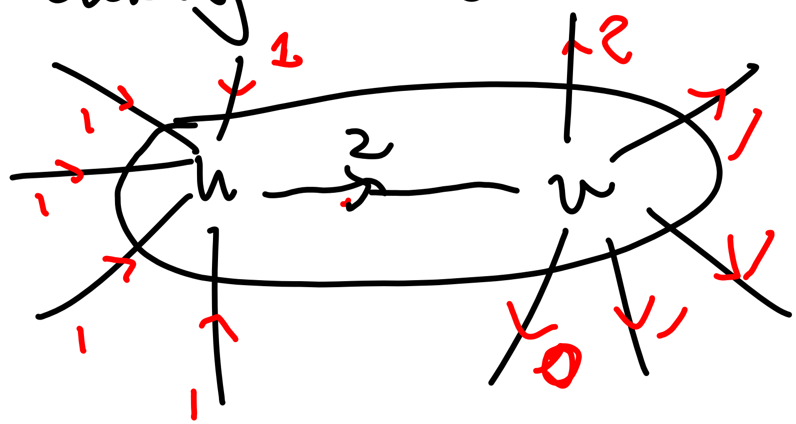
Adding edge e does not increase effective resistance.
If added edge is not incident to a then addition does not decrease the escape prob. $= \frac{1}{c(a)\, R(a \leftrightarrow z)}$

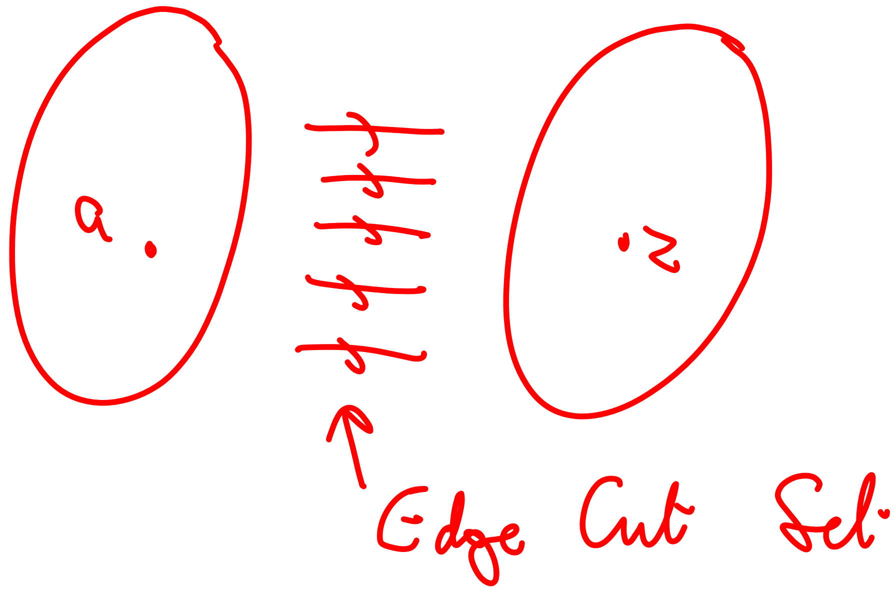
Cor. 9.14

Gluing does not increase effective resistance.

Pf

Inf in Thomsons theorem is over a larger class of flows





Prop 9.15 (Nash-Williams)

If $\{\Pi_k\}$ are disjoint edge cut-sets

then

$$R(a \leftrightarrow z) \geq \sum_k \left(\sum_{e \in \Pi_k} c(e) \right)^{-1}$$

Pf Fix k:

$$\sum_{e \in \Pi_k} c(e) \sum_{e \in \Pi_k} r(e)\,\theta(e)^2 \geq \left(\sum_{e \in \Pi_k} \sqrt{c(e)}\sqrt{r(e)}\,|\theta(e)| \right)^2$$

$$= \left(\sum_{e \in \Pi_k} |\theta(e)| \right)^2 \geq \|\theta\|^2$$

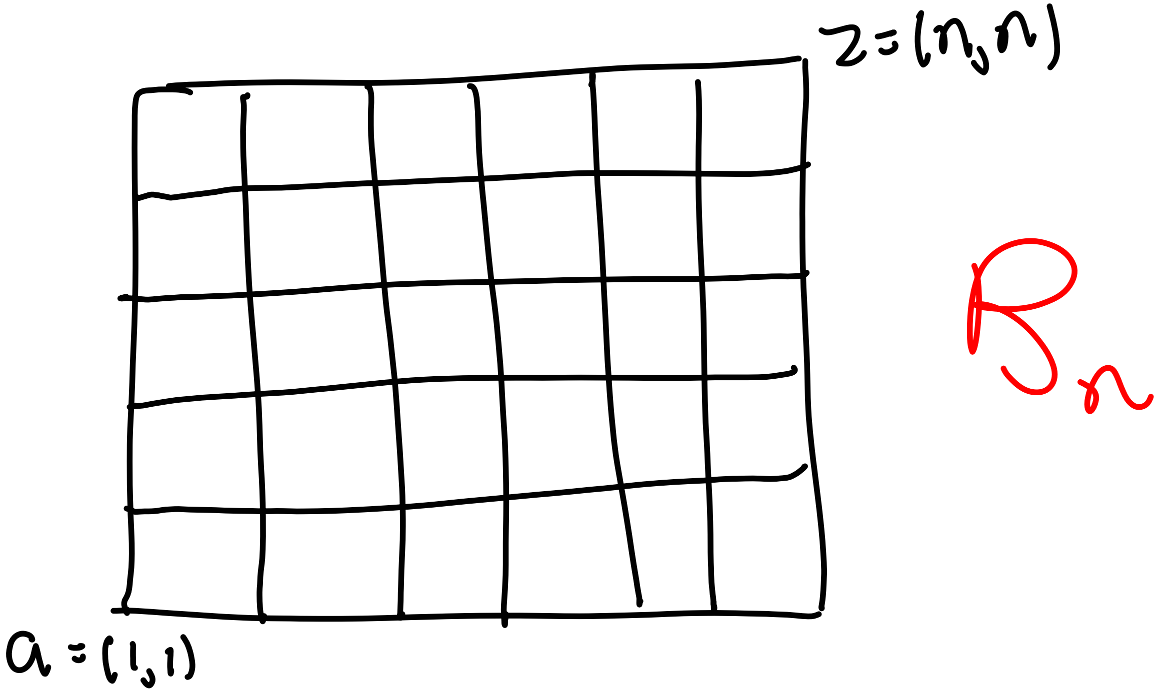
$$\geq 1$$

$$\sum_e r(e)\,\theta(e)^2 \geq \sum_k \sum_{e \in \Pi_k} r(e)\,\theta(e)^2$$

$$\geq \sum_k \left(\sum_{e \in \Pi_k} c(e) \right)^{-1}$$

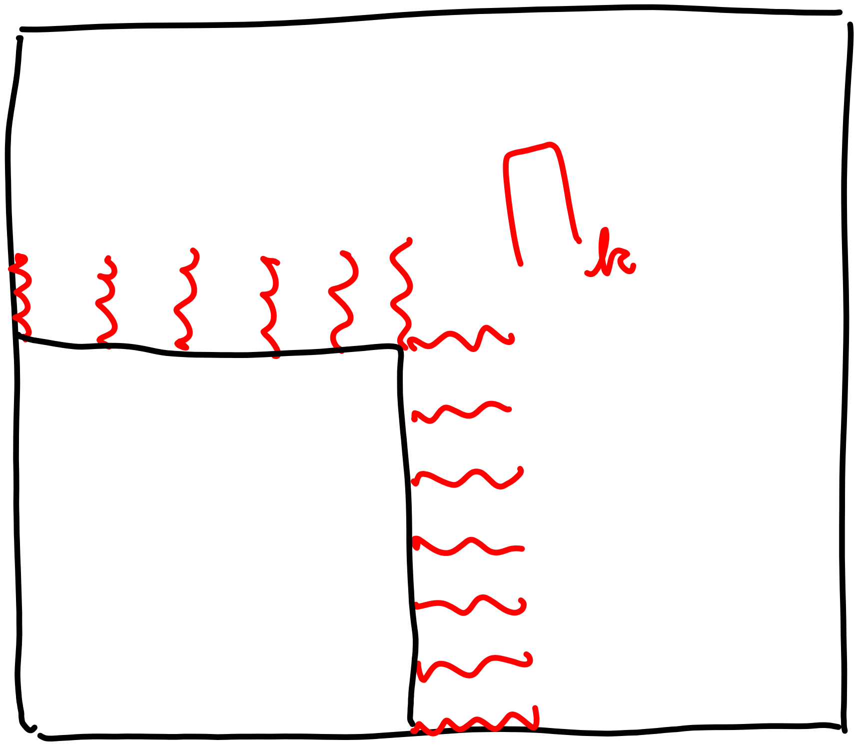
Apply Thomson

Escape on square



$$\frac{\log(n-1)}{2} \leq R(a \leftrightarrow z) \lesssim 2\log n.$$

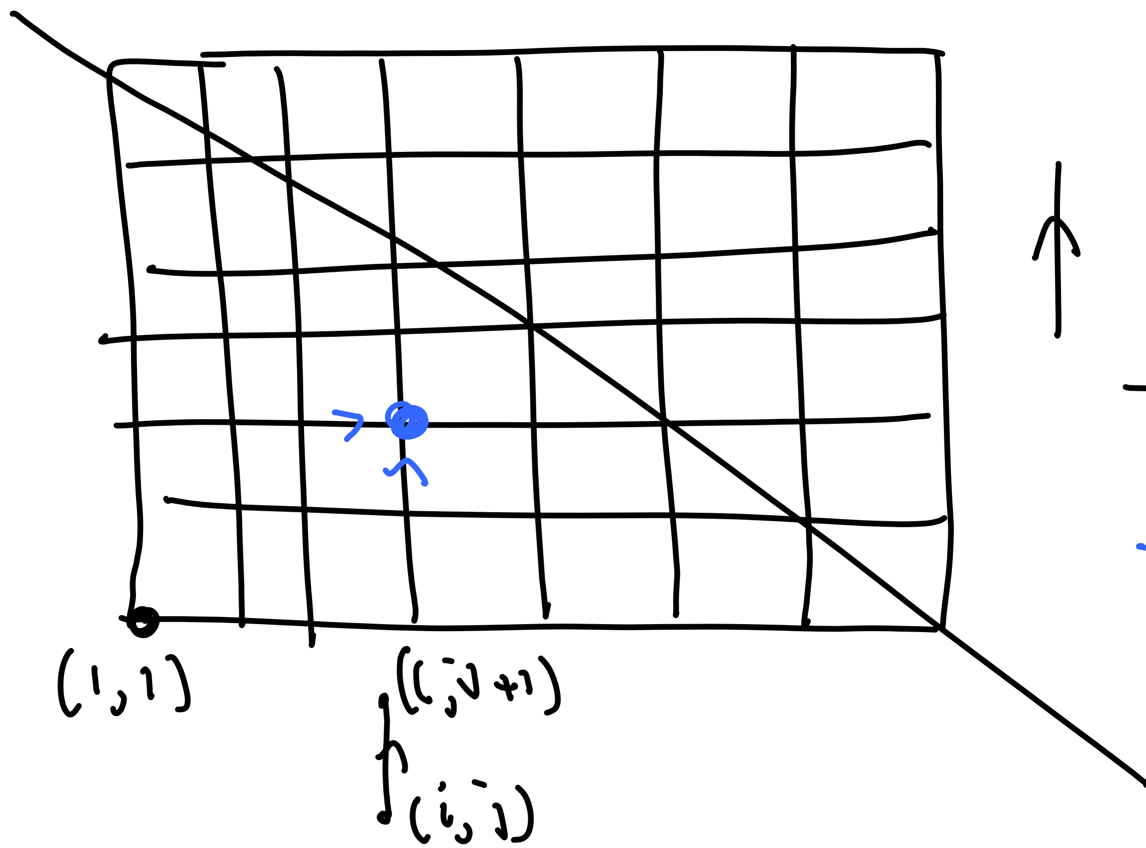
Lower Bound



$$\Pi_k = \{ (v,w) \in B_n : \|v\|_\infty = k-1, \ \|w\|_\infty = k \}$$

$$\sum_{e \in \Pi_k} c(e) = 2k \ ; \quad R \geq \sum_{k=1}^{n-1} \frac{1}{2k}$$

Upper Bound



$$f(e_1)^2 + f(e_2)^2 \le \frac{1}{(k+1)^2}$$

$$f(e_1) + f(e_2) = \frac{1}{k+1}$$

f(e) = P[Polya urn process goes through e]

$$E(f) \le 2\sum_{k=1}^{n-1} (k+1)\cdot\frac{1}{(k+1)^2} \le 2\log n$$