3\29\10

Effective Resistance

$$R(a \leftrightarrow z) = \frac{W(a) - W(z)}{\| I \|}$$

Indendendent of W. – get same value for any choice of voltage.

P9.5

For any $a, z \in \Omega$

$$P_a(\tau_z < \tau_a^+) = \frac{1}{c(a)\, R(a \leftrightarrow z)}$$

Probability that z is visited before a return to a – escape probability.

Proof

Function

$$x \longmapsto E_x \mathbb{1}_{\{X_{\tau(a,z)} = z\}} = P_x(\tau_z < \tau_a)$$

↑ time to visit a or z

Proof

Function

$$x \longmapsto E_x \mathbb{1}_{\{X_{\underset{\substack{\uparrow \\ \text{time to visit} \\ a \text{ or } z}}{\tau(a,z)}} = z\}} = P_x(T_z < T_a)$$

is the unique function that is harmonic on $\Omega \setminus \{a, z\}$ and takes value 0 at a & 1 at z.

But

$$x \longmapsto \frac{W(a) - W(x)}{W(a) - W(z)}$$

satisfies

Uses harmonicity of W.

So,

$$P_x\{\tau_z < \tau_a\} = \frac{W(a)-W(x)}{W(a)-W(z)}.$$

So

$$P_a(\tau_z < \tau_a^+) = \sum_{x\in V} P(a,x)\,P_x(\tau_z < \tau_a)$$

$$= \sum_{x\sim a} \frac{c(a,x)}{c(a)} \cdot \frac{W(a)-W(x)}{W(a)-W(z)}$$

$$= \sum_{x\sim a} \frac{I(\overrightarrow{ax})}{c(a)\left(W(a)-W(z)\right)}$$

$$= \sum_{x \sim a} \frac{I(\overrightarrow{ax})}{c(a)\left(W(a)-W(z)\right)}$$

$$P_a(\tau_z < \tau_a^+) = \frac{\| I \|}{c(a)\left(W(a)-W(z)\right)}.$$

$$= \frac{1}{c(a)\, R(a \leftarrow z)}$$

Green's Function

τ is a stopping time.

$$G_\tau(a,x) = E_a(\#\text{visits to } x \text{ before } \tau).$$

L9.6

$$G_{\tau_z}(a,a) = c(a)\, R(a \leftrightarrow z)$$

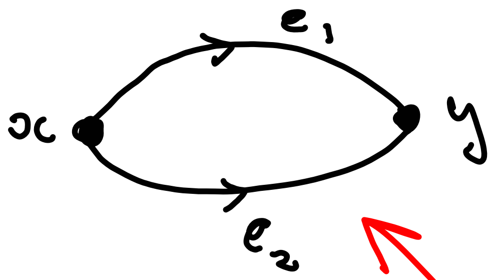
E[# of returns to a before visiting z]

Proof

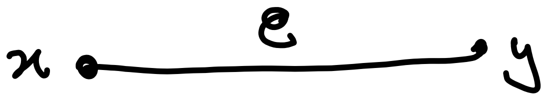
visits to a is a geometric random variable with parameter $P_a(T_z < \tau_a^+) = \frac{1}{c(a)\,R(a \leftrightarrow z)}$.

Network Simplification

Parallel Law



$\equiv$



$c(e) = c(e_1) + c(e_2)$

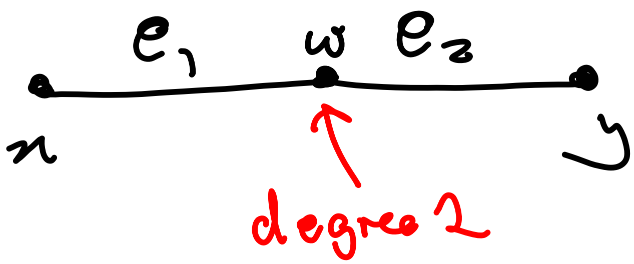
Check:

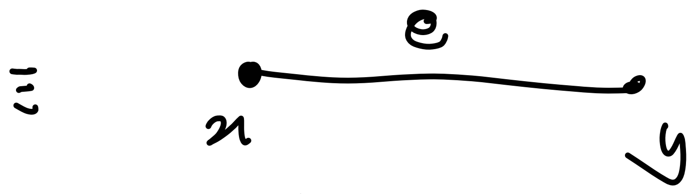
Ohm's Law: $I(e_j) = [W(y) - W(x)]\, c(e_j)$

$I(e) = [W(y) - W(x)]\,(c(e_1) + c(e_2)]$

Kirchoff law satisfied.

Series Law





$r(e) = r(e_1) + r(e_2)$

Check: $\frac{[W(x) - W(w)]}{r(e_1)} = I(e_1)$ $\qquad \frac{W(x) - W(y)}{r(e)} = I(e)$

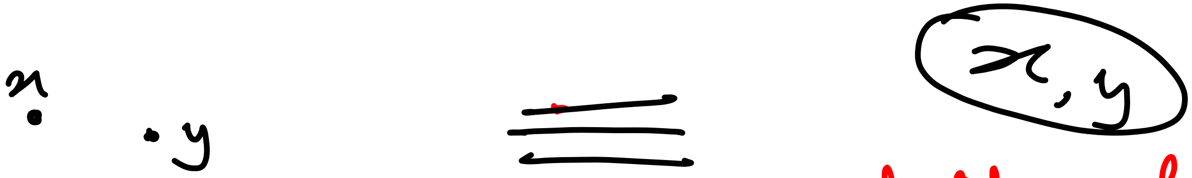
$\|$

$= \frac{[W(w) - W(y)]}{r(e_2)} = I(e_2)$

$$r(e) = \frac{W(x) - W(y)}{I(e)} = \frac{W(x) - W(w)}{I(e_1)} + \frac{W(w) - W(y)}{I(e_2)}$$

$I(e) = I(e_1) = I(e_2)$

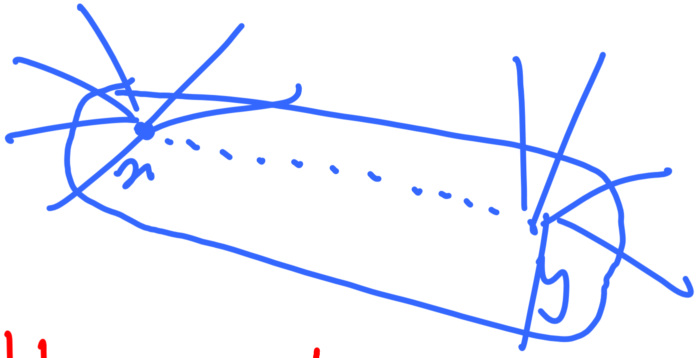
Gluing



Same voltage

$W(x) = W(y)$

identify vertices
and keep edge.



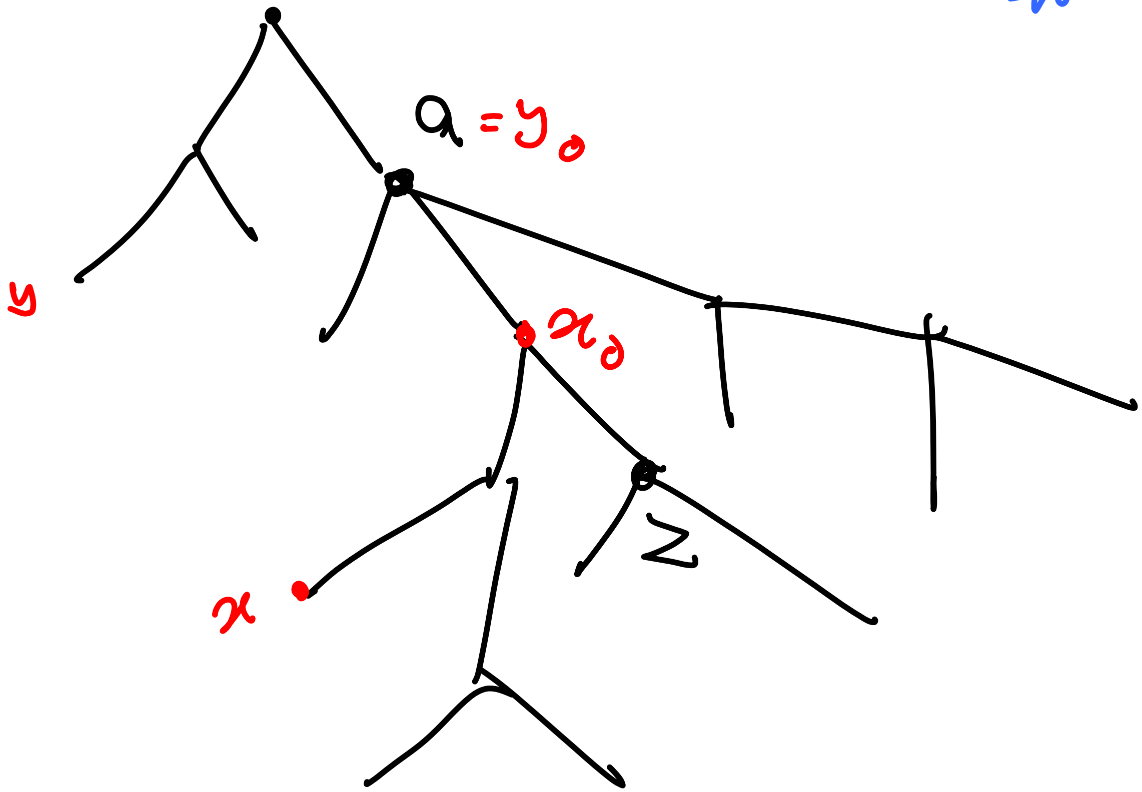
Voltages unchanged

$\Rightarrow$ current unchanged.

Ex 9.7

Path a to z = $\overset{a}{a_0}, a_1, \cdots, a_k$

$W(a_i) = 1 - \frac{i}{k}$



(i) $R(a \leftrightarrow z) = d(a, z)$

(ii) $W(x) = W(x_0)$