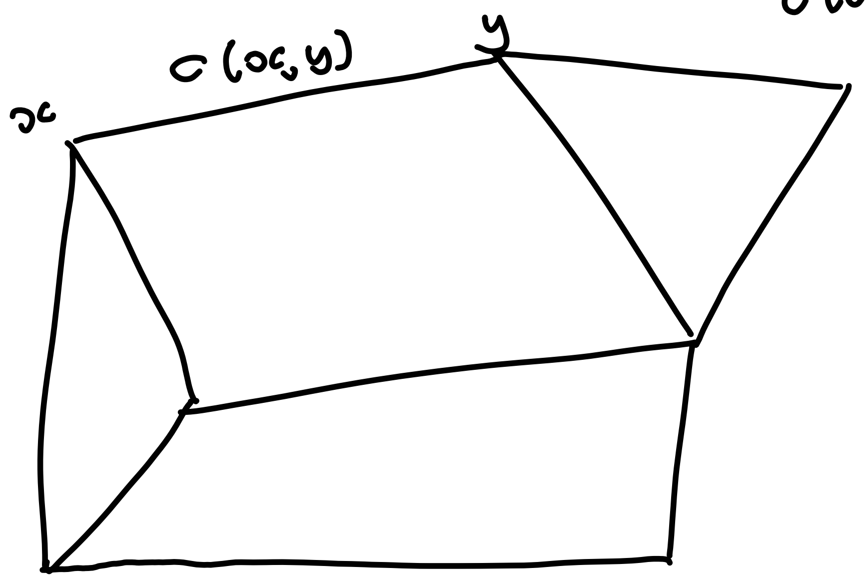
3/24/10

Networks

$e = (x, y)$

$c(e) = c(x,y) =$ conductance

$r(e) = \frac{1}{c(e)} =$ resistance



Markov chain:

$$P(x,y) = \frac{c(x,y)}{c(x)} \qquad c(x) = \sum_y c(x,y)$$

Reversible: $\pi(x) = \frac{c(x)}{c_G}$ Detailed Balance $c_G = \sum_x c(x)$

Markov chain $\equiv$ simple random walk on multi-graph.

Conversely: If P corresponds to a reversible Markov chain,

$$c(x,y) = \pi(x)\, P(x,y)$$

yields a network

$$c(x) = \pi(x) \quad \text{and} \quad c_G = 1$$

$$\frac{c(x,y)}{c(x)} = P(x,y)$$

Recall $h : \Omega \to \mathbb{R}$ is harmonic for $x \in \Omega$ if

$\Omega = V$

$$h(x) = \sum_{y \in \Omega} P(x,y)\, h(y)$$

[If h is harmonic everywhere $h = Ph$]

$B \subseteq \Omega$. $\tau_B = \min\{t \geq 0 : X_t \in B\}$

P 9.1

Given $\left\{ \begin{array}{l} (X_t) = \text{chain with irreducible } P \\ B \subsetneq \Omega \qquad h_B : B \longrightarrow \mathbb{R} \end{array} \right.$

Claim: the function $h : \Omega \longrightarrow \mathbb{R}$ defined by

$$h(x) = \mathbb{E}_x\, h_B(X_{\tau_B})$$

is the unique extension of h_B to Ω such that (i) $h(x) = h_B(x)$, $x \in B$ and (ii) h is harmonic at $x \notin B$.

Pf

Suppose $x \notin B$. Then $\tau_B \geq 1$.

$$h(x) = \sum_y P(x,y) \underbrace{E_x\left[h(X_{\tau_B}) \,\middle|\, X_1 = y\right]}_{E_y(h(X_{\tau_B}))}$$

$$= \sum_y P(x,y)\, h(y)$$

$\Rightarrow$ h is harmonic at $x \notin B$

Uniqueness :

Suppose there are two possibilities h_1, h_2.

Write $g = h_1 - h_2$.

(i) $g(x) = 0, x \in B$, (ii) g is harmonic for $x \notin B$

To show $g(x) = 0, x \notin B$.

Suppose $\exists x \notin B$ such that $g(x) > 0$

and let

$$A = \{ x : g(x) = \max_{B^c} g \}$$

Suppose $x \in A$ and $P(x,y) > 0$ then we must have $y \in A$.

$$g(x) = \sum_{z \in \Omega} g(z) P(x,z) \qquad \text{harmonic}$$

max

average of quantities $\leq$ max

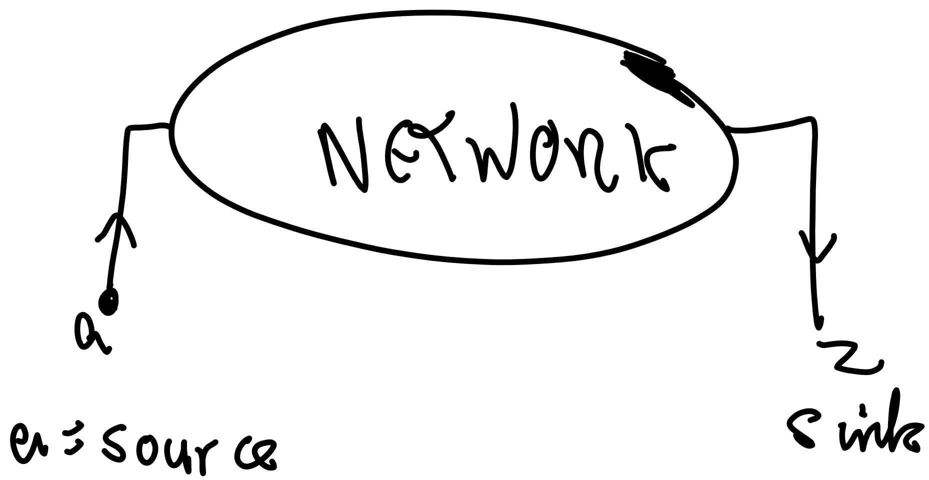
$\Rightarrow$ all equal max.

This implies that $A = \Omega$

So $g(x) \leq 0$, $x \notin B$

and similarly $g(x) \geq 0$, $x \notin B$.

Voltage



W is harmonic on $V \setminus \{a, z\}$ is a voltage

i.e. fix $W(a)$, $W(z)$ and apply 9.1.

Flow θ

$e = \{x, y\}$ $\quad\quad$ $\vec{e} = (x, y)$

(i) $\theta(\overrightarrow{xy}) = -\theta(\overrightarrow{yx})$

(ii) $\operatorname{div} \theta(x) = \sum_{y \sim x} \theta(\overrightarrow{xy})$

(a) $\operatorname{div} \theta(x) = 0, \quad x \neq a, z$ $\quad$ Kirchoff Law

(b) $\operatorname{div} \theta(a) \geq 0$ $\quad$ $\operatorname{div} \theta(z) := -\operatorname{div} \theta(a).$

$$\sum_{x \in \Omega} \operatorname{div} \theta(x) = \sum_{e} \left[\theta(\overrightarrow{xy}) + \theta(\overrightarrow{yx})\right] = 0$$

Strength of θ, $\|\theta\| = \operatorname{div} \theta(a)$

Unit flow $\|\theta\| = 1$

Given W: current flow

$$I(\overrightarrow{xy}) = \frac{W(x) - W(y)}{r(x,y)}$$

Ohm's Law

$x \notin \{a, z\}$

$$\sum_{y \sim x} I(\overrightarrow{xy}) = \sum_{y \sim x} \underbrace{c(x,y)} (W(x) - W(y))$$

$$= c(x) W(x) - \sum_{y \sim x} \underbrace{c(x) P(x,y)} W(y)$$

$$= 0, \quad \text{(W is harmonic).}$$

I is a flow.

Cycle Law

Suppose $(x_0, x_1, \ldots, x_{k-1})$ is a cycle.

$$\vec{e}_i = x_{i-1}x_i$$

Then

$$\sum_{i=1}^{k} r(\vec{e}_i)\, I(\vec{e}_i) = \sum_{i=1}^{k} \left[W(x_i) - W(x_{i-1}) \right] = 0$$

total voltage change

I obeys cycle law.

P 9.4

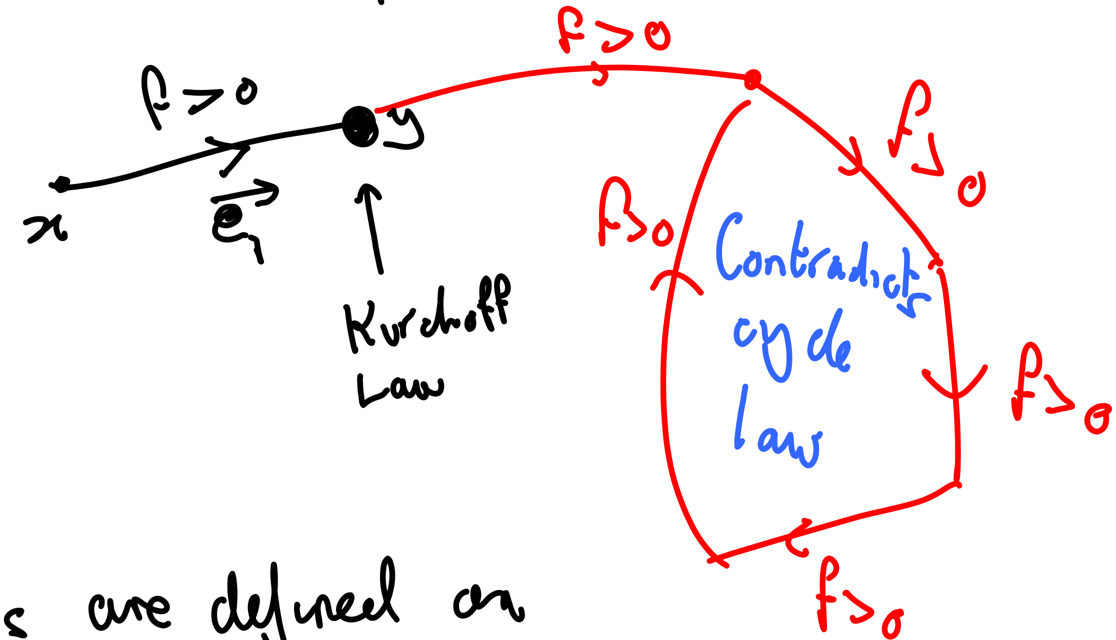
If θ is a flow that obeys cycle law and $\|\theta\| = \|I\|$ then $\theta = I$.

Proof

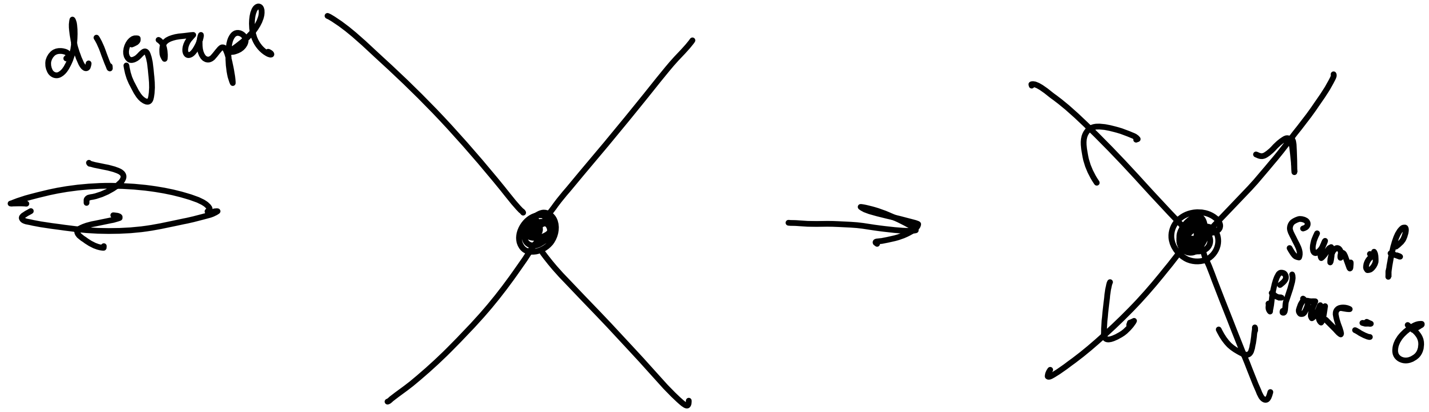
$f = \theta - I$ satisfies Kirchoff law at every vertex (including a & z) and the cycle law.

Must show $f = 0$.

Suppose $f(\vec{e}_1) > 0$



flows are defined on digraph



Effective Resistance

$$R(a \leftrightarrow z) = \frac{W(a) - W(z)}{\|I\|}$$

Independent of W.
(Same as independent of
$W(a)$, $W(z)$)

(i) Adding λ to both sides does not change R.

(ii) Scaling by λ does not change R.

Can assume $W(a) = 1$
$W(z) = 0$