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Thm 4.9

Suppose that P is irreducible and aperiodic then $\exists\ \alpha \in (0,1)$ and $C > 0$ such that

$$\| P^t(x, \cdot) - \pi \|_{TV} \leq C \alpha^t$$

Proof

$\exists\ r > 0$ such that $P^r(x,y) > 0$,

$\forall x, y$.

Choose δ such that

$$P^r(x,y) \geq \delta \pi(y) \quad , \quad \forall x,y$$

Let $\Pi = \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}$ and write

$$P^r = (1-\theta)\Pi + \theta Q$$ define Q

Q is stochastic i.e. $Q\mathbf{1} = \mathbf{1}$

$$\theta Q \mathbf{1} = P^r \mathbf{1} - (1-\theta)\Pi\mathbf{1} = \theta \mathbf{1}$$

$M\Pi = \Pi$ if M is stochastic

$= 1$ $\begin{bmatrix} \pi_1 \\ \pi_1 \\ \vdots \\ \pi_1 \end{bmatrix}$

and $\Pi M = \Pi$ if $\pi M = \pi$

<u>Claim</u> $P^{rk} = (1-\theta^k)\Pi + \theta^k Q^k$

<u>Proof</u> Induction on k.

$k=1$, by definition of Q

$$p^{r(k+1)} = \left((1-\theta^{k})\Pi + \theta^{k} Q^{k}\right) p^{r}$$

inductive assumption

$$= (1-\theta^{k})\underbrace{\Pi p^{r}}_{\Pi} + \theta^{k} Q^{k} p^{r}$$

$$= (1-\theta^{k})\Pi + \theta^{k} Q^{k} p^{r}$$

$$= (1-\theta^{k})\Pi + \theta^{k} Q^{k}\left((1-\theta)\Pi + \theta Q\right)$$

$$= \Pi - \theta^{k}\Pi + \theta^{k}(1-\theta)\Pi + \theta^{k+1} Q^{k+1}$$

Q^{k} is stochastic

$$= \Pi(1-\theta^{k+1}) + \theta^{k+1} Q^{k+1}.$$

$\square$

$$P^{rk+j} - \Pi = \left((1-\theta^k)\Pi + \theta^k Q^k\right)P^j - \Pi$$

$$= (1-\theta^k)\Pi + \theta^k Q^k P^j - \Pi$$

$$= \theta^k \left(\underbrace{Q^k P^j}_{\text{stochastic}} - \Pi\right)$$

A, B stoch $\Downarrow$ AB stoch.

$AB\mathbb{1} = A\mathbb{1} = \mathbb{1}$

Fix row x_0

$$\| P^{rk+j}(x_0, \cdot) - \pi \|_{TV} \leq \theta^k$$

$\square$

4.4

$$d(t) = \max_{x \in \Omega} \| P^t(x, \cdot) - \pi \|_{TV}$$

$$\bar{d}(t) = \max_{x, y} \| P^t(x, \cdot) - P^t(y, \cdot) \|_{TV}$$

Lemma 4.11

$$d(t) \leq \bar{d}(t) \leq 2\, d(t)$$

Proof

$$\bar{d}(t) \leq \max_{x, y} \left[\| P^t(x, \cdot) - \pi \|_{TV} + \| \pi - P^t(y, \cdot) \|_{TV} \right]$$

$$\leq 2\, d(t)$$

$$\pi(A) = \sum_{x \in A} \left[\sum_{y \in \Omega} \pi(y) P^t(y, x) \right]$$

$$= \sum_{y \in \Omega} \pi(y) P^t(y, A)$$

So

$$\| P^t(x, \cdot) - \pi \|_{TV} = \max_A | P^t(x, A) - \pi(A) |$$

$$= \max_A \left| \sum_{y \in \Omega} \pi(y) \left(P^t(x, A) - P^t(y, A) \right) \right|$$

$$\leq \max_A \sum_{y \in \Omega} \pi(y) \left| P^t(x, A) - P^t(y, A) \right|$$

$$\leq \max_{A} \sum_{y \in \Omega} \pi(y) \left| P^t(x, A) - P^t(y, A) \right|$$

$$\leq \sum_{y \in \Omega} \pi(y) \max_{A} \left| P^t(x, A) - P^t(y, A) \right|$$

$$= \sum_{y \in \Omega} \pi(y) \left\| P^t(x, \cdot) - P^t(y, \cdot) \right\|_{TV}$$

$$\leq \max_{y} \left\| P^t(x, \cdot) - P^t(y, \cdot) \right\|_{TV}$$

$$\leq \bar{d}(t)$$

$$\Rightarrow d(t) \leq \bar{d}(t).$$

Claim

(i) $$d(t) = \sup_{\mu \in \mathcal{P}} \|\mu P^t - \pi\|_{TV}$$

($\mathcal{P}$:= distributions on Ω)

(ii) $$\bar{d}(t) = \sup_{\mu, \nu \in \mathcal{P}} \|\mu P^t - \nu P^t\|_{TV}$$

Lemma 4.2

$$\bar{d}(s+t) \leq \bar{d}(s)\,\bar{d}(t)$$

Proof

Fix $x, y \in \Omega$ and let X_s, Y_s be an optimal coupling of $P^s(x,\cdot)$, $P^s(y,\cdot)$

$$\|P^s(x,\cdot) - P^s(y,\cdot)\|_{TV} = P(X_s \neq Y_s)$$

$$P^{s+t}(x,w) = \sum_z P^s(x,z)\,P^t(z,w)$$

$$= \sum_z P[X_s = z]\,P^t(z,w) = E\left(P^t(X_s, w)\right)$$

So,

$$P^{s+t}(x,w) - P^{s+t}(y,w) = E\left[P^t(X_s,w) - P^t(Y_s,w)\right]$$

and

$$\|P^{s+t}(x,\cdot) - P^{s+t}(y,\cdot)\|_{TV}$$

$$= \frac{1}{2}\sum_w \left| E\left(P^t(X_s,w) - P^t(Y_s,w)\right)\right|$$

$$\leq E\left(\frac{1}{2}\sum_w \left|P^t(X_s,w) - P^t(Y_s,w)\right|\right)$$

$|E(X)| \leq E(|X|)$ – Jensen's Ineq.

$$= E\left(\|P^t(X_s,\cdot) - P^t(Y_s,\cdot)\|_{TV}\right)$$

$$= E\left(\| P^t(X_s, \cdot) - P^t(Y_s, \cdot) \|_{TV} \right)$$

$$\leq E\left(\bar{d}(t) \cdot \mathbb{1}_{X_s \neq Y_s} \right)$$

$$= \bar{d}(t) \, P_r\left(X_s \neq Y_s \right)$$

maximise over x, y.

$$\leq \bar{d}(t) \, \bar{d}(s).$$

$\square$

Ex 4.3 $\quad \|\mu P - \nu P\|_{TV} \leq \|\mu - \nu\|_{TV}$

$\mu = P^t(x,\cdot)$ & $\nu = \pi$

$$\|P^{t+1}(x,\cdot) - \pi\|_{TV} \leq \|P^t(x,\cdot) - \pi\|_{TV}$$

$d(t) \searrow$ and similarly $\bar{d}(t) \searrow$

$$|\mu P(A) - \nu P(A)| = \left|\sum_{y\in A}\sum_{x\in\Omega} \mu_x P_{x,y} - \sum_{y\in A}\sum_{x\in\Omega} \nu_x P_{x,y}\right|$$

$$= \left|\sum_{x\in\Omega} (\mu_x - \nu_x) P_{x,A}\right|$$

$$\leq \sum_{\substack{x\in\Omega \\ \mu_x \geq \nu_x}} (\mu_x - \nu_x) = \|\mu - \nu\|_{TV}.$$

Let c positive integer

$$d(ct) \leq \bar{d}(ct) \leq \bar{d}(t)^c$$

Mixing Time

$$t_{mix}(\epsilon) = \min\{t : d(t) \leq \epsilon\}$$

$$t_{mix} = t_{mix}(1/4)$$

something $< \frac{1}{2}$

$$d(\ell\, t_{mix}(\epsilon)) \leq \bar{d}(\ell\, t_{mix}(\epsilon))$$

$$\leq \bar{d}(t_{mix}(\epsilon))^{\ell}$$

$\bar{d} \leq 2d$

$$\leq (2\epsilon)^{\ell}$$

Ergodic Theorem: P irreducible

$f : \Omega \to \mathbb{R}$

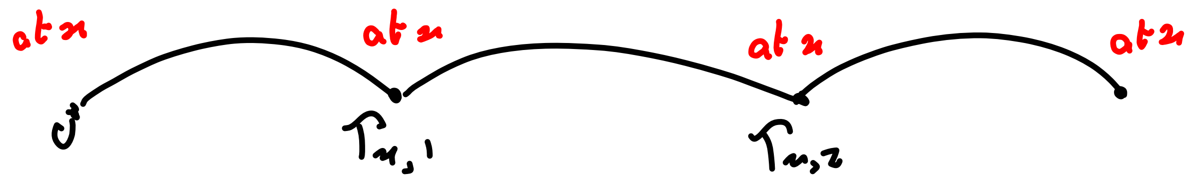
$\mu \in \mathcal{P}$, starting distribution

$$P_\mu\left\{ \lim_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t-1} f(X_s) - \sum_x \pi_x f(x) \right\} = 1$$

Proof

Suppose chain starts at x.

$$\tau^+_{x,0} = 0 \quad \text{and} \quad \tau^+_{x,k} = \min\{ t > \tau^+_{x,k-1} : X_t = x \}$$



$$Y_k = \sum_{s=\tau^+_{x,k-1}}^{\tau^+_{x,k}-1} f(X_s)$$

is a sequence of i.i.d. random variables

If

$$S_t = \sum_{s=0}^{t-1} f(X_s)$$

then

$$S_{\tau^+_{x,n}} = \sum_{k=1}^{n} Y_k$$

So, by law of large numbers

$$P_x\left\{ \lim_{n\to\infty} \frac{S_{\tau^+_{x,n}}}{n} = E_x(Y_1) \right\}$$

Now

$$T^+_{x,n} = \sum_{k=1}^{n} \underbrace{\left(T^+_{x,k} - T^+_{x,k-1} \right)}_{\text{i.i.d.}}$$

$$P_x \left\{ \lim_{n\to\infty} \frac{T^+_{x,n}}{n} = E_x\left(T^+_x\right) \right\}$$

$$P_x \left\{ \lim_{n\to\infty} \frac{S_{T^+_{x,n}}}{T^+_{x,n}} = \frac{E_x(Y_1)}{E_x\left(T^+_x\right)} \right\} = 1$$

Now

$$E_x(Y_1) = E_x\left(\sum_{s=0}^{T_x^+-1} f(X_s)\right)$$

$$= E_x\left(\sum_{y\in\Omega} f(y) \sum_{s=0}^{T_x^+-1} 1_{X_s=y}\right)$$

$$= \sum_{y\in\Omega} f(y)\, E_x\left(\sum_{s=0}^{T_x^+-1} 1_{X_s=y}\right)$$

$$\pi_y = \frac{E_x(\quad)}{E(T_x^+)}$$

$$= \sum_{y\in\Omega} f(y)\,\pi(y)\, E(T_x^+)$$

So

$$P_x\left\{\lim_{n\to\infty} \frac{S_{\tau^+_{x,n}}}{\tau^+_{x,n}} = E_\pi(f)\right\} = 1$$

Ex 4.7 $a_n < \infty$, $n_k \to \infty$; $\frac{n_{k+1}}{n_k} \to 1$; $\frac{a_1 + \cdots + a_{n_k}}{n_k} \to a$ $\Rightarrow$ $\frac{a_1 + \cdots + a_n}{n} \to a$

Let $n_k = \sum_{j \le k} \tau^+_{x,j}$. $P_r\left\{\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = 1\right\} = 1$

So $P_x\left\{\lim_{t\to\infty} \frac{S_t}{t} = E_\pi(f)\right\} = 1$

We need to show that $P_r\left\{\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = 1\right\} = 1$

This is implied by $P_r\left\{\lim_{k\to\infty} \frac{\tau^+_{x,k}}{k} = 0\right\} = 1$

$n_k \geq k$

This is implied by $P_r\left\{\lim_{k\to\infty} \frac{\tau^+_{x,k}}{k} = 0\right\} = 1$ $\leftarrow n_k \geq k$

Choose $\epsilon > \delta, r$ such that $p^r(y,x) \geq \delta > 0, \ \forall y \neq r.$

Then

$$P_r\left\{\frac{\tau^+_{x,k}}{k} \geq \epsilon\right\} \leq (1-\delta)^{\epsilon k/r}$$

and

$$P_r\left\{\lim_{k\to\infty} \frac{\tau^+_{x,k}}{k} \geq \epsilon\right\} > 0$$

$$\Rightarrow \underbrace{P_r\left\{\bigcap_{k=1}^{\infty} \bigcup_{j\geq k} \left\{\frac{\tau^+_{x,k}}{k} \geq \epsilon\right\}\right\}}_{\leq \lim_{k\to\infty} \sum_{j\geq k} (1-\delta)^{\epsilon j/r} = 0.} > 0$$