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Thm 4.9

Suppose that P is irreducible and aperiodic then $\exists \alpha \in (0,1)$ and $C > 0$ such that

$$\|P^t(\alpha, \cdot) - \pi\|_{TV} \leq C\alpha^t$$

Proof

$\exists r > 0$ such that $P^r(\alpha, y) > 0$,

$\forall \alpha, y$.

Choose S such that

$$P^r(x,y) \geq S\pi(y), \quad \forall x,y$$

Let $\Pi = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_n \end{bmatrix}$ and write

$$P^r = (1-\theta)\Pi + \theta Q \quad \text{defines } Q$$

Q is stochastic i.e. $Q_{11} = 1$

$$\theta Q_{11} = P^r_{11} - (1-\theta) \Pi_{11} = \theta 1$$

$$M \sqcap = \sqcap \text{ if } M \text{ is stochastic}$$

A diagram illustrating the multiplication of a stochastic matrix P by itself k times to produce a stochastic matrix M . On the left, there is a wavy line labeled $=1$ above it. Above the wavy line is a curved arrow pointing from P to M . To the right of the wavy line is a large square bracket containing several smaller square brackets, each labeled with a π above it, representing probability vectors.

and

$$\sqcap^k M = \sqcap \text{ if } \pi M \leq \pi$$

Claim $P^{\tau_k} = (1 - \theta^k) \sqcap + \theta^k Q^k$

Proof Induction on k .

$k=1$, by definition of Q

$$\begin{aligned}
 P^r(k+1) &= ((1 - \theta^k) \sqcap + \theta^k Q^k) P^r && \text{Inductive assumption} \\
 &= (1 - \theta^k) \underbrace{\sqcap}_{\text{P}} P^r + \theta^k Q^k P^r \\
 &= (1 - \theta^k) \sqcap + \theta^k Q^k P^r \\
 &= (1 - \theta^k) \sqcap + \theta^k Q^k ((1 - \theta) \sqcap + \theta Q) \\
 &= \sqcap - \theta^k \sqcap + \theta^k (1 - \theta) \sqcap + \theta^{k+1} Q^{k+1} \\
 &\quad \text{Q^k is stochastic} \\
 &= \sqcap \left(-\theta^{k+1} \right) + \theta^{k+1} Q^{k+1}.
 \end{aligned}$$

□

$$\begin{aligned}
 P^{rk+j} - \pi &= ((1-\theta^k)\pi + \theta^k Q^k) P^j - \pi \\
 &= (1-\theta^k)\pi + \theta^k Q^{rk} P^j - \pi \\
 &= \theta^k (Q^{rk} P^j - \pi)
 \end{aligned}$$

stochastic

Fix row x_0

$$\|P^{rk+j}(x_0, \cdot) - \pi\|_{TV} \leq \theta^k$$

A, B stochastic
 \Downarrow
AB stochastic.
AD1, AI = 1

□

4.4

$$d(t) = \max_{x \in \mathcal{X}} \|P^t(x, \cdot) - \pi\|_{TV}$$

$$\bar{d}(t) = \max_{x, y} \|P^t(x, \cdot) - P^t(y, \cdot)\|_{TV}$$

Lemma 4.11

$$d(t) \leq \bar{d}(t) \leq 2 d(t)$$

Proof

$$\begin{aligned}\bar{d}(t) &\leq \max_{x, y} \left[\|P^t(x, \cdot) - \pi\|_{TV} + \|\pi - P^t(y, \cdot)\|_{TV} \right] \\ &\leq 2 d(t)\end{aligned}$$

$$\pi(A) = \sum_{x \in A} \left[\sum_{y \in \Omega} \pi(y) P^t(y, x) \right]$$

$$= \sum_{y \in \Omega} \pi(y) P^t(y, A)$$

So

$$\|P^t(x, \cdot) - \pi\|_{TV} = \max_A |P^t(x, A) - \pi(A)|$$

$$= \max_A \left| \sum_{y \in \Omega} \pi(y) (P^t(x, A) - P^t(y, A)) \right|$$

$$\leq \max_A \sum_{y \in \Omega} \pi(y) |P^t(x, A) - P^t(y, A)|$$

$$\leq \max_A \sum_{y \in Q} \pi(y) |P^t(x, A) - P^t(y, A)|$$

$$\leq \sum_{y \in Q} \pi(y) \max_A |P^t(x, A) - P^t(y, A)|$$

$$= \sum_{y \in Q} \pi(y) \|P^t(x, \cdot) - P^t(y, \cdot)\|_{TV}$$

$$\leq \max_y \|P^t(x, \cdot) - P^t(y, \cdot)\|_{TV}$$

$$\leq \bar{d}(t)$$

$$\Rightarrow d(t) \leq \bar{d}(t).$$

Claim

$$(i) \quad d(t) = \sup_{\mu \in P} \|\mu P^t - \pi\|_{TV}$$

(P : distributions on Ω)

$$(ii) \quad \bar{d}(t) = \sup_{\mu, \nu \in P} \|\mu P^t - \nu P^t\|_{TV}$$

Lemma 4.2

$$\bar{d}(s+t) \leq \bar{d}(s) \bar{d}(t)$$

Proof

Fix $x, y \in \mathcal{R}$ and let X_s, Y_s be an optimal coupling of $P^s(x, \cdot), P^s(y, \cdot)$

$$\|P^s(x, \cdot) - P^s(y, \cdot)\|_{TV} = P_1(X_s \neq Y_s)$$

$$P^{s+t}(x, w) = \sum_z P^s(x, z) P^t(z, w)$$

$$= \sum_z P[X_s = z] P^t(z, w) = E(P^t(X_s, w))$$

$$\text{So, } P^{s+t}(x_s, \omega) - P^{s+t}(y_s, \omega) = E \left[P^t(X_s, \omega) - P^t(Y_s, \omega) \right]$$

and

$$\|P^{s+t}(x_s, \cdot) - P^{s+t}(y_s, \cdot)\|_{TV} = \frac{1}{2} \sum_w |E(P^t(X_s, \omega) - P^t(Y_s, \omega))|$$

$$\leq E \left(\frac{1}{2} \sum_w |P^t(X_s, \omega) - P^t(Y_s, \omega)| \right)$$

$\therefore \uparrow |E(X)| \leq E(|X|) - \text{Jensen's Ineq.}$

$$= E \left(\|P^t(X_s, \cdot) - P^t(Y_s, \cdot)\|_{TV} \right)$$

$$= E \left(\| P^t(X_s, \cdot) - P^t(Y_s, \cdot) \|_{TV} \right)$$

$$\leq E \left(\bar{d}(t) \cdot \mathbb{1}_{X_s \neq Y_s} \right)$$

$$= \bar{d}(t) P(X_s \neq Y_s)$$

maximize
over x, y .

$$\leq \bar{d}(t) \bar{d}(s).$$

□

E4.3

$$\|\mu P - \nu P\|_{TV} \leq \|\mu - \nu\|_{TV}$$

$$\mu = P^t(x, \cdot) \text{ & } \nu = \pi$$

$$\|P^{t+1}(x, \cdot) - \pi\|_{TV} \leq \|P^t(x, \cdot) - \pi\|_{TV}$$

$d(t) \downarrow$ and similarly $\bar{d}(t) \downarrow$

$$|\mu P(A) - \nu P(A)| = \left| \sum_{y \in A} \sum_{x \in \Omega} \mu_x P_{x,y} - \sum_{y \in A} \sum_{x \in \Omega} \nu_x P_{x,y} \right|$$

$$= \left| \sum_{x \in \Omega} (\mu_x - \nu_x) P_{x,A} \right|$$

$$\leq \sum_{x \in \Omega} |\mu_x - \nu_x| = \|\mu - \nu\|_{TV}.$$

$$\mu_x \geq \nu_x$$

Let positive integer

$$d(ct) \leq \bar{d}(ct) \leq \bar{d}(t)^c$$

Mixing Time

$$t_{\max}(\epsilon) = \min \{ t : d(t) \leq \epsilon \}$$

$$t_{\max} = t_{\max}(1/4)$$

something
 $\approx \frac{1}{2}$

$$\begin{aligned} d(l t_{\max}(\epsilon)) &\leq \bar{d}(l t_{\max}(\epsilon)) \\ &\leq \bar{d}(t_{\max}(\epsilon))^l \quad \bar{d} \leq 2d \\ &\approx (2\epsilon)^l \end{aligned}$$

Ergodic Theorem: P irreducible

$$f: \Omega \rightarrow \mathbb{R}$$

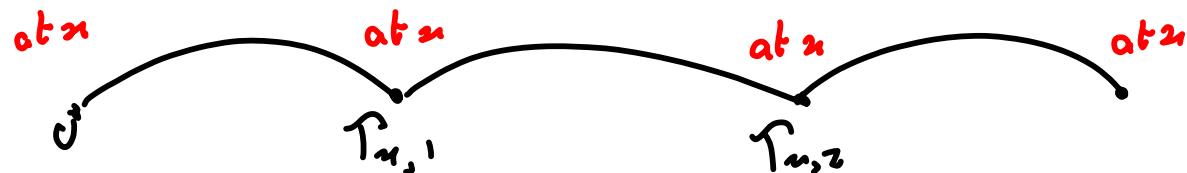
$\mu \in \mathcal{P}$, starting distribution

$$P_\mu \left\{ \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{s=0}^{t-1} f(X_s) - \sum_n \pi_n f(x_n) \right\} = 1$$

Proof

Suppose chain starts at x .

$$T_{x,0}^+ = 0 \text{ and } T_{x,t_0}^+ = \min \left\{ t > T_{x,t_0-1}^+ : X_t = x \right\}$$



$$Y_k = \sum_{s=0}^{T_{x,k}-1} f(X_s)$$

is a sequence of i.i.d. random variables

If

$$S_x = \sum_{s=0}^{b-1} f(X_s)$$

then

$$S_{\sum_{i \in N} T_{x,i}} = \sum_{k=1}^n Y_k$$

So, by law of large numbers

$$P_x \left\{ \lim_{n \rightarrow \infty} \frac{S_{\sum_{i \in N} T_{x,i}}}{n} = E_x(Y_1) \right\}$$

Now

$$\mathcal{T}_{x,n}^+ = \sum_{k=1}^n \left(\mathcal{T}_{x,k}^+ - \mathcal{T}_{x,k-1}^+ \right)$$

i.i.d.

$$P_n \left\{ \lim_{n \rightarrow \infty} \frac{\mathcal{T}_{x,n}^+}{n} < E_n(\mathcal{T}_n^+) \right\}$$

$$P_n \left\{ \lim_{n \rightarrow \infty} \frac{\frac{S \mathcal{T}_{x,n}^+}{\mathcal{T}_{x,n}^+}}{\mathcal{T}_{x,n}^+} = \frac{E_n(Y_1)}{E_n(\mathcal{T}_n^+)} \right\} = 1$$

Now

$$E_x(Y_1) = E_x \left(\sum_{s=0}^{T_n^+ - 1} f(X_s) \right)$$

$$= E_x \left(\sum_{y \in \Omega} f(y) \sum_{s=0}^{T_n^+ - 1} 1_{X_s=y} \right)$$

$$= \sum_{y \in \Omega} f(y) E_x \left(\sum_{s=0}^{T_n^+ - 1} 1_{X_s=y} \right)$$

$$\pi_y \frac{E_x(\cdot)}{E(T_n^+)}$$

$$= \sum_{y \in \Omega} f(y) \pi(y) E(T_n^+)$$

S.

$$P_n \left\{ \lim_{n \rightarrow \infty} \frac{\sum \tau_{x_0, n}^+}{\tau_{x_0, n}^+} = E_\pi(\beta) \right\} = 1$$

Ex 4.) $a_n < \infty, n_k \rightarrow \infty ; \frac{n_{k+1}}{n_k} \rightarrow 1 ; \frac{a_1 + \dots + a_{n_k}}{n_k} \rightarrow a \Rightarrow \frac{a_1 + \dots + a_n}{n} \rightarrow a$

Let $n_k = \sum_{j \leq k} \tau_{x_0, j}^+$. $P_r \left\{ \lim_{k \rightarrow \infty} \frac{n_{k+1}}{n_k} = 1 \right\} = 1$

so $P_\pi \left\{ \lim_{t \rightarrow \infty} \frac{S_t}{t} = E_\pi(f) \right\} = 1$

We need to show that $P_r \left\{ \lim_{k \rightarrow \infty} \frac{n_{k+1}}{n_k} = 1 \right\} = 1$

This is implied by

$$P_r \left\{ \lim_{k \rightarrow \infty} \frac{\tau_{x_0, k}^+}{k} = 0 \right\} = 1$$

$\nwarrow n_k \geq k$

This is implied by $\Pr_r \left\{ \lim_{k \rightarrow \infty} \frac{T_{x,k}^+}{k} = 0 \right\} = 1$

$\nwarrow n_k > k$

Choose $\epsilon > \delta, r$ such that $\Pr_r(y, z) \geq \delta > 0, \forall y \neq r$.

Then

$$\Pr_r \left\{ \frac{T_{x,k}^+}{k} \geq \epsilon \right\} \leq (1 - \delta)^{\epsilon k/r}$$

and

$$\Pr_r \left\{ \lim_{k \rightarrow \infty} \frac{T_{x,k}^+}{k} > \epsilon \right\} \geq \theta$$

$$\Rightarrow \Pr_r \left\{ \bigcap_{k=1}^{\infty} \bigcup_{j \geq k} \left\{ \frac{T_{x,j}^+}{j} \geq \epsilon \right\} \right\} \geq \theta$$

$\leq \lim_{k \rightarrow \infty} \sum_{j \geq k} (1 - \delta)^{\epsilon j/r} = 0.$