2/24/2010

Chapter 4: Mixing

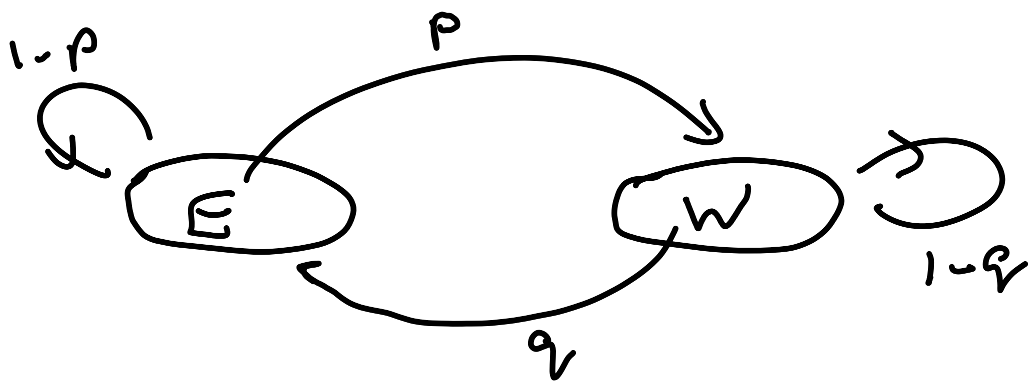
Total Variation Distance

Let μ, ν be 2 distributions on Ω.

$$\|\mu - \nu\|_{TV} = \max_{A \subseteq \Omega} |\mu(A) - \nu(A)|$$

Ex 1

Frog:



Steady State: $\pi = \left(\frac{q}{p+q}, \frac{p}{p+q} \right)$

Events: $A = \emptyset, \Omega, \{E\}, \{W\}$

μ_t = distribution after t steps

$$\Delta_t = \underbrace{\mu_t(E) - \pi(E)}_{\|\mu_t - \pi\|} = \underset{\Downarrow\, 0}{(1-p-q)^t}\, \underset{\uparrow\, \frac{p}{p+q}}{\Delta_0}$$

Prop 4.2

$$\|\mu - \nu\|_{TV} = \frac{1}{2}\sum_{x\in\Omega} |\mu(x) - \nu(x)|$$

$$= \sum_{\substack{x\in\Omega \\ \mu(x)\geq\nu(x)}} (\mu(x) - \nu(x))$$

$B \leftarrow \{x\in\Omega,\ \mu(x)\geq\nu(x)\}$

Proof

$A \subseteq \Omega$. $\mu(A) - \nu(A) \leq \mu(A\cap B) - \nu(A\cap B)$

$$\leq \mu(B) - \nu(B)$$

$$\nu(A) - \mu(A) \leq \nu(B^c) - \mu(B^c)$$

$$= \mu(B) - \nu(B)$$

$\square$

P4.4

$$\|\mu - \nu\|_{TV} \leq \|\mu - \eta\|_{TV} + \|\eta - \nu\|_{TV}$$

P4.5

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sup_{\|f\| \leq 1} \left\{ \sum_{x \in \Omega} \mu(x) f(x) - \sum_{x \in \Omega} \nu(x) f(x) \right\}$$

Proof

$$\frac{1}{2}\left(\sum_{x \in \Omega} \mu(x) f(x) - \sum_{x \in \Omega} \nu(x) f(x)\right) \leq \frac{1}{2} \sum_{x \in \Omega} |f(x)| \cdot |\mu(x) - \nu(x)|$$

$$\leq \|\mu - \nu\|_{TV}$$

Now let:

$$f^*(x) = \begin{cases} 1 & : \mu(x) \geq \nu(x) \\ -1 & : \mu(x) < \nu(x) \end{cases}$$

$$\frac{1}{2}\left(\sum_{x \in \Omega} f^*(x)\mu(x) - \sum_{x \in \Omega} f^*(x)\nu(x)\right)$$

$$= \frac{1}{2}\left(\mu(B) - \mu(B^c) - (\nu(B) - \nu(B^c))\right)$$

$$= \mu(B) - \nu(B)$$

$$= \|\mu - \nu\|_{TV}$$

4.2 Coupling

A coupling of distributions μ, ν on Ω is a pair of random variables (X, Y) such that

$$\Pr(X=x) = \sum_{y \in \Omega} \underbrace{\Pr(X=x, Y=y)}_{q(x,y)} = \mu(x)$$

marginal

$$\Pr(Y=y) = \sum_{x} q(x,y) = \nu(y)$$

Simple Example

$$\Omega = \{H, T\}$$

(I)
$$\begin{array}{c|cc} & H & T \\ H & \frac{1}{4} & \frac{1}{4} \\ T & \frac{1}{4} & \frac{1}{4} \end{array}$$

$$Q = q(x,y)$$

X & Y independent

(II)
$$\begin{array}{c|cc} & H & T \\ H & \frac{1}{2} & \\ T & & \frac{1}{2} \end{array}$$

$$X = Y$$

In general
$$\begin{array}{c|cc} & H & T \\ H & \alpha & \frac{1}{2}-\alpha \\ T & \frac{1}{2}-\alpha & \alpha \end{array}$$

Coupling Lemma

P4.7

$$\|\mu-\nu\|_{TV} = \inf\left[P[X\neq Y] : (X,Y) \text{ is a coupling of } \mu,\nu\right]$$

Proof

Fix $A \subseteq \Omega$

$$\mu(A)-\nu(A) = P[X\in A] - P[Y\in A] = \sum_{\substack{x\in A\\ y\in\Omega}} q(x,y) - \sum_{\substack{x\in\Omega\\ y\in A}} q(x,y)$$

$$\mu(A) - \nu(A) = P[X \in A] - P[Y \in A] = \sum_{\substack{x \in A \\ y \in \Omega}} q(x,y) - \sum_{\substack{x \in \Omega \\ y \in A}} q(x,y)$$

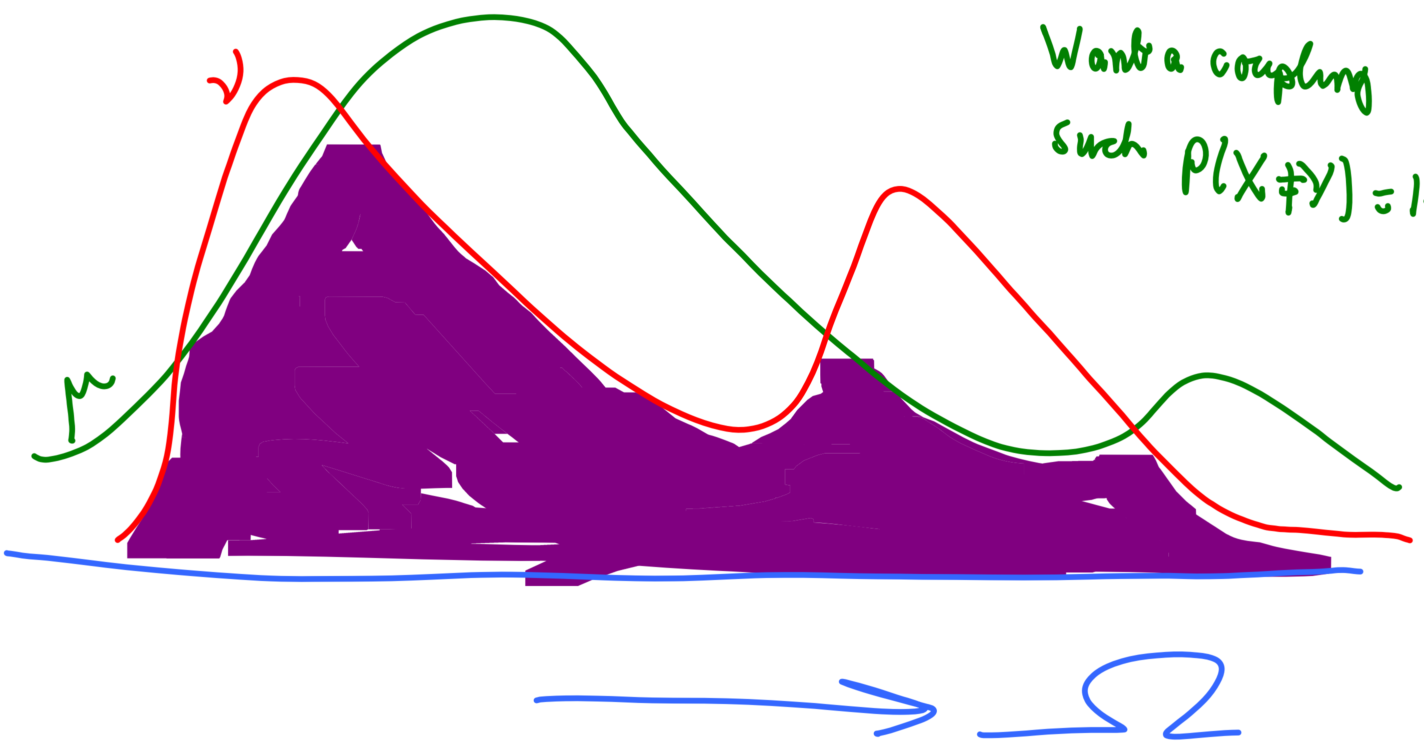
$$\leq \sum_{\substack{x \in A \\ y \in \Omega}} q(x,y) - \sum_{\substack{x \in A \\ y \in A}} q(x,y)$$

$$= \sum_{\substack{x \in A \\ y \notin A}} q(x,y)$$

$$\leq P[X \neq Y]$$

$\| \cdots \|_{TV} \leq \inf$

Now we want the maximal coupling.



Want a coupling such $P(X \neq Y) = 1 - p$

$$p = \sum_{x \in \Omega} \min\{\mu(x), \nu(x)\} = 1 - \|\mu - \nu\|_{TV}$$

$$\sum_{x \in B} \nu(x) + \sum_{x \in B^c} \mu(x) = \nu(B) + 1 - \mu(B)$$

$\wedge \equiv \min$

With probability p: $X = Y = x$ according to $\dfrac{\mu(x) \wedge \nu(x)}{p}$

$\xi^+ := \max\{0, \xi\}$

With probability $1-p$:

$X \neq Y$ $\left\{ \begin{array}{l} X = x, \text{ prob } \dfrac{(\mu(x) - \nu(x))^+}{1-p} = \gamma_1(x) \\ Y = y, \text{ prob } \dfrac{(\nu(y) - \mu(y))^+}{1-p} = \gamma_2(y) \end{array} \right.$

$$P[X=x] = p \times \frac{\mu(x) \wedge \nu(x)}{p} + (1-p) \frac{(\mu(x)-\nu(x))^+}{1-p}$$

$$= \mu(x)$$

$$P[Y=y] = \cdot \qquad = \nu(y)$$

$$P[X \neq Y] = 1-p$$

$$= \|\mu - \nu\|_{TV}$$