1/27/10

L2.18

(a) $P_k(T_0 < r, X_r = j) = P_k(X_r = -j)$

(b) $P_k(T_0 < r, X_r > 0) = P_k(X_r < 0)$.

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Ex: $P_1(T_0 = 2m+1) = P_1(T_0 > 2m, X_{2m} = 1, X_{2m+1} = 0)$

$= P_1(T_0 > 2m, X_{2m} = 1) \times \frac{1}{2}$

So $P_1(T_0 = 2m+1) = \frac{1}{2}\left[P_1(X_{2m} = 1) - P_1(T_0 \leq 2m, X_{2m} = 1)\right]$

$= \frac{1}{2}\left[P_1(X_{2m} = 1) - P_1(X_{2m} = -1)\right]$

$$= \frac{1}{2}\left[P_1(X_{2m}=1) - P_1(X_{2m}=-1)\right]$$

$$= \frac{1}{2}\left[\binom{2m}{m} \times \frac{1}{2^{2m}} - \binom{2m}{m-1} \times \frac{1}{2^{2m}}\right]$$

$$= \frac{1}{2^{2m+1}} \cdot \frac{1}{m+1} \cdot \binom{2m}{m}$$

<u>L 2.21</u>

$$P_k(\tau_0 > r) = P_0(-k < X_r \leq k)$$

<u>Pf</u>

$$P_k(X_r > 0) = P_k(X_r > 0, \tau_0 \leq r) + P_r(\tau_0 > r)$$

$$= P_k(X_r > 0, \tau_0 < r) + P_r(\tau_0 > r)$$

$$= P_k(X_r > 0, T_0 < r) + P_r(T_0 > r)$$

$$= P_k(X_r < 0) + P_k(T_0 > r)$$

$$= P_k(X_r > 2k) + P_k(T_0 > r)$$

So

$$P_k(T_0 > r) = P_k(X_r > 0) - P_r(X_r > 2k)$$

$$= P_k(0 < X_r \leq 2k)$$

$$= P_0(-k < X_0 \leq k)$$

$\blacksquare$

$\underline{L2.22}$ $$P_0(X_t = k) \leq \frac{3}{\sqrt{t}}$$

$\underline{Pf}$

$$P_0[X_{2r} = 2k] = \binom{2r}{r+k}\left(\frac{1}{2}\right)^{2r}$$

$$\leq \binom{2r}{r}\left(\frac{1}{2}\right)^{2r}$$

$$= \frac{(2r)!}{r!^2} \cdot \left(\frac{1}{2}\right)^{2r}$$

$$\leq \frac{\sqrt{4\pi r}}{2\pi r} \cdot 2$$

$$\leq \frac{2}{\sqrt{\pi r}}$$

Stirling's Formula

$$m! \approx \sqrt{2\pi m}\left(\frac{m}{e}\right)^m$$

$$\leq e^{\frac{1}{12m}}$$

$$\geq$$

$$P_0(X_{2r+1} = 2k+1) \leq \frac{3}{\sqrt{\pi r}}$$

Pf of Thm 2.11]

$$P_k(T_0 > r) = P_0(-k < X_r \leq k)$$

$$= \sum_{t=1-k}^{k} P_0(X_r = t)$$

$$\leq 2k \cdot \frac{3}{\sqrt{\pi r}}$$

$$= O\left(\frac{k}{\sqrt{r}}\right).$$

## Thm 2.26

Let $\{\Delta_i\}$ be i.i.d. integer valued variables with mean zero and variance $\sigma^2$. Then

identically independently

$$\underbrace{P[X_t \neq 0 \text{ for } 1 \le t \le r]}_{\alpha_r} \le \frac{4\sigma}{\sqrt{r}}.$$

**Proof**

$I \subseteq \mathbb{Z}$

$$L_r(I) = \{t \in \{0,1,\dots,r\} : X_t \in I\}$$

= times of visit to $I$.

$$A_r = \{t \in L_r(0) : X_{t+u} \neq 0 \text{ for } 1 \le u \le r\}$$

times of visits to 0 without return in $r$ steps

$$P(t \in A_r) = P(t \in L_r(0))\, P_\theta\underbrace{(X_t \neq 0,\ 1 \le u \le r)}_{\alpha_r}$$

Summing

$$1 \ge E(|A_r|) = \underbrace{E(|L_r(0)|)}_{\text{Need a lower bound on this to upper bound } \alpha_r}\, \alpha_r$$

By Chebyshev,

Let $X_t = \Delta_1 + \Delta_2 + \cdots + \Delta_t$

$$P(|X_t| \ge \sigma\sqrt{r}) \le \frac{E(X_t^2)}{\sigma^2 r} = \frac{t\sigma^2}{\sigma^2 r} = t/r$$

So if $I = (-\sigma\sqrt{r}, +\sigma\sqrt{r})$

$$E(|L_r(I^c)|) = \sum_{t=1}^{r} P(X_t \notin I)$$

$$\leq \sum_{t=1}^{r} \frac{t}{r} = \frac{r+1}{2}$$

and so

$$E(|L_r(I)|) \geq r+1 - \frac{r+1}{2} \geq \frac{r}{2}$$

For any $v \neq 0$

$$\underbrace{E(|L_r(v)|)} = E\left(\sum_{t=0}^{r} \mathbb{1}_{\{X_t = v\}}\right)$$

$$= E\left(\sum_{t=\tau_v}^{r} \mathbb{1}_{\{X_t = v\}}\right)$$

$$\leq E_v\left(\sum_{t=0}^{r} \mathbb{1}_{\{X_t = v\}}\right)$$

$$= E_0\left(\sum_{t=0}^{r} \mathbb{1}_{\{X_t = 0\}}\right)$$

$$= E_0(|L_r(0)|)$$

Sum over $v \in I$

So

$$r/2 \leq E(|L_r(I)|) \leq |I| \times E_0(|L_r(0)|)$$

$$\leq 20\sqrt{r}\, E_0(|L_r(0)|)$$

and we have $\alpha_r \lesssim 1/E_0(|L_r(0)|)$.