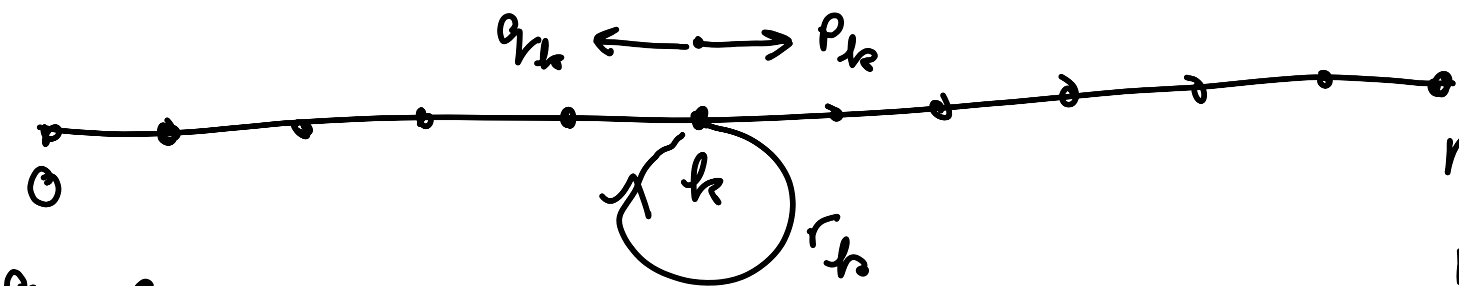
1\25\10

Birth and death chain



$q_0 = 0$ $\qquad$ $p_n = 0$

Let $\qquad w_k = \prod_{i=1}^{k} \frac{p_{i-1}}{q_i} \qquad w_0 = 1$

$k \geq 1$

$$p_{k-1} w_{k-1} = q_k w_k$$

$P[k-1 \to k]$ $\qquad$ $P[k \to k-1]$

$w_0, w_1, \dots w_k$ satisfies detailed balance

So

(i) Chain is reversible

(ii) $\pi_k \propto \omega_k$ so $\pi_k = \dfrac{\omega_k}{\omega_0 + \cdots + \omega_k}$

proportional

Fix $\ell \in \{0, 1, \ldots, n\}$ and restrict chain

to $L := \{0, 1, 2, \ldots, \ell\}$

Same moves as before, except that

$p_\ell, q_\ell, r_\ell \longrightarrow 0, q_\ell, p_\ell + r_\ell$

check balance

$\tilde{\pi}_k \equiv$ steady state for new chain $= \dfrac{\omega_k}{\omega_0 + \cdots + \omega_\ell}$

$\tilde{E}$ = expectation w.r.t. this chain.

Hitting Times

$$\tilde{E}_l(\tau_l^+) = 1 + q_l \underbrace{\tilde{E}_{l-1}(\tau_l)}_{= E_{l-1}(\tau_l)}$$

$$\tilde{E}_l(\tau_l^+) = 1/\tilde{\pi}_l = \frac{w_0 + w_1 + \cdots + w_l}{w_l}$$

$$E_{l-1}(\tau_l) = \frac{1}{q_l} \sum_{j=0}^{l-1} w_j$$

$$E_a(\tau_b) = \sum_{l=a+1}^{b} E_{l-1}(\tau_l).$$

$a < b$

Special Case: $(p_k, q_k, r_k) = (p, q, r)$

$k \neq 0, n$

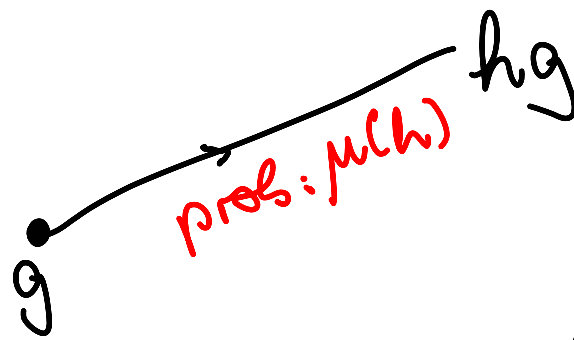
$p \neq q$: $\omega_k = \left(\frac{p}{q}\right)^k, \quad 0 \leq k \leq n$

$$E_{l-1}(\tau_l) = \frac{1}{q(p/q)^l} \sum_{j=0}^{l-1} (p/q)^j = \frac{1}{p-q}\left[1-\left(\frac{q}{p}\right)^l\right]$$

$p = q$: $\omega_k = 1$ $\qquad E_{l-1}(\tau_l) = \frac{l}{p}$

Random Walks on Groups

G is a finite group.

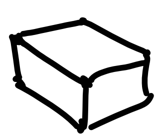


μ is a measure on G.

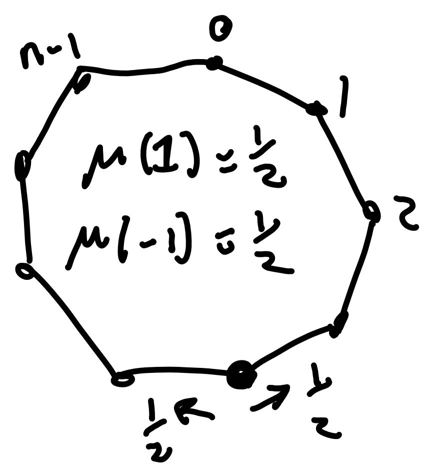
$$\sum_{h \in G} \mu(h) = 1$$

Cube : $G = \mathbb{Z}_2^n$ $\quad e_1, e_2, \ldots, e_n$

$\mu = \frac{1}{n} \quad \frac{1}{n} \quad \ldots \quad \frac{1}{n}$



Cycle $G = \mathbb{Z}_n$



Prop 2.12

P is transition matrix of random walk on group G. Then U = uniform distribution is stationary.

Proof

Detailed balance fails

$$U(g)\underbrace{P(g,hg)}_{\mu(h)} \stackrel{?}{=} U(hg)\underbrace{P(hg,g)}_{\mu(h^{-1})}$$

(1) $\mu(h) = \mu(h^{-1}) \Rightarrow$ (i) $U(g)$ is stationary

(ii) Chain is reversible.

Otherwise chain is not reversible.

$$\sum_{h \in G} U(h)\, P(h, g) = \frac{1}{|G|} \sum_{k \in G} P(\underbrace{k^{-1} g}_{h}, g)$$

$$\Vert ?$$

$$U(g)$$

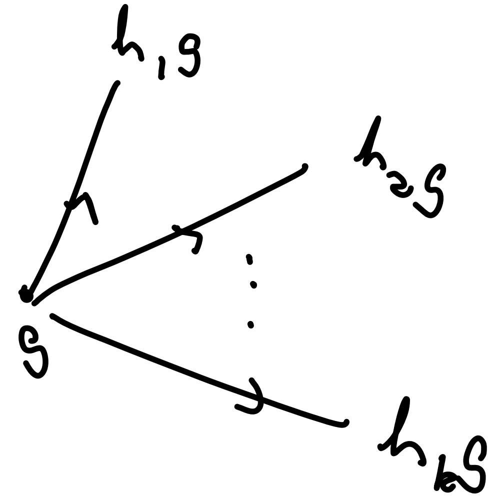
$$= \frac{1}{|G|} \sum_{k \in G} \mu(k)$$

2.6.1)

Generating sets, irreducibility, reversibility.

If $H \subseteq G$ then $\langle H \rangle :=$ subgroup generated by H

H is a set of generators if $\langle H \rangle = G$



$\langle h_1, \dots, h_k \rangle = G$

Cayley Graph

Prop 2.13

Random walk is irreducible $\iff \langle \overbrace{\{h : \mu(h) > 0\}}^{H} \rangle = G$.

Proof

$\Rightarrow$

$\exists k: \quad p^k(id, a) > 0 \Rightarrow \exists\, a_1, a_2, \dots, a_k \in H$

s.t. $a = a_k a_{k-1} \cdots a_1$

$\Leftarrow$ suppose $a, b \in G$. Can write $ba^{-1} = s_r s_{r-1} \cdots s_1$

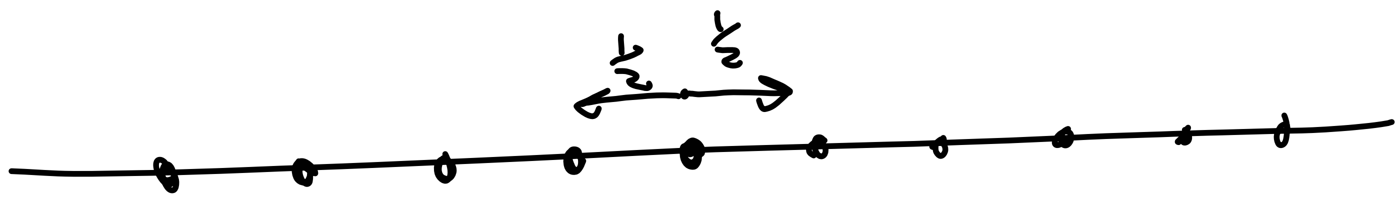
$p^r(a,b) \geq p(a, s_1 a)\, p(s_1 a, s_2 s_1 a) \cdots \qquad s_i \in H$

Walk is symmetric if $\mu(g) = \mu(g^{-1})$ $\forall g$

$\Downarrow$

Reversible

2.7 Random walks on $\mathbb{Z}$ and Reflection principle.



Thm 2.10

$$P_k(T_0 > r) \leq \frac{12k}{\sqrt{r}}$$

L2.18 (Reflection Principle)

(a) $$P_k(T_0 < r, X_r = j) = P_k(X_r = -j)$$

b) $$P_k(T_0 < r, X_r > 0) = P_r(X_r < 0)$$

Proof

(a) $\Longrightarrow$ (b) : sum over $j > 0$.

$$P_k(T_0 = s, X_r = j) = P_k(T_0 = s)\, P_0(X_{r-s} = j).$$

$s < r$ and $j > 0$

// reflection

$$= P_k(T_0 = s)\, P_0(X_{r-s} = -j)$$

Sum over $s < r$: $P_k(T_0 < r, X_r = j) = P_k(X_r = -j)$

Remember $k > 0$