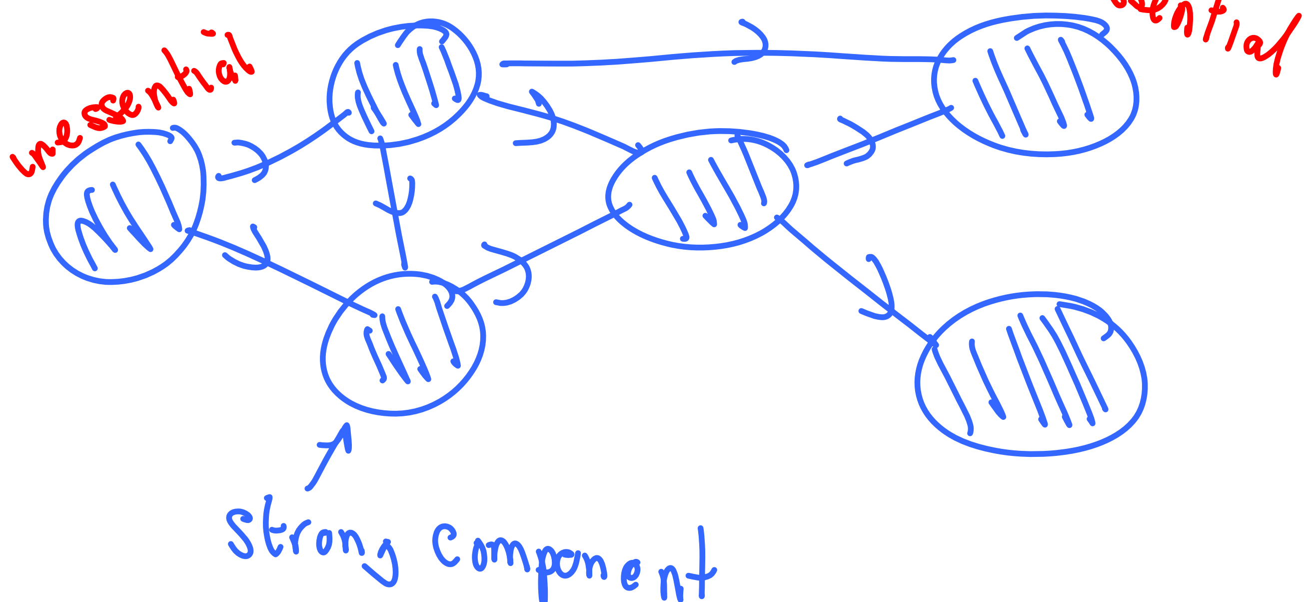
1\20\10

1.7

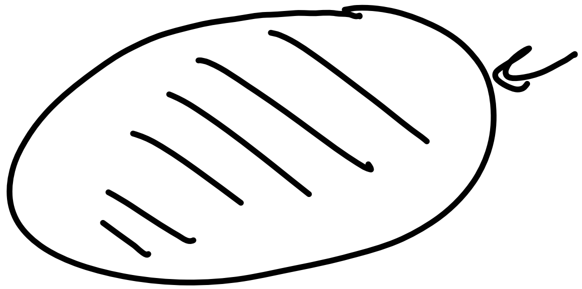
$x \in \Omega$ is essential if

$x \to y$ implies $y \to x$

∃ path x to y

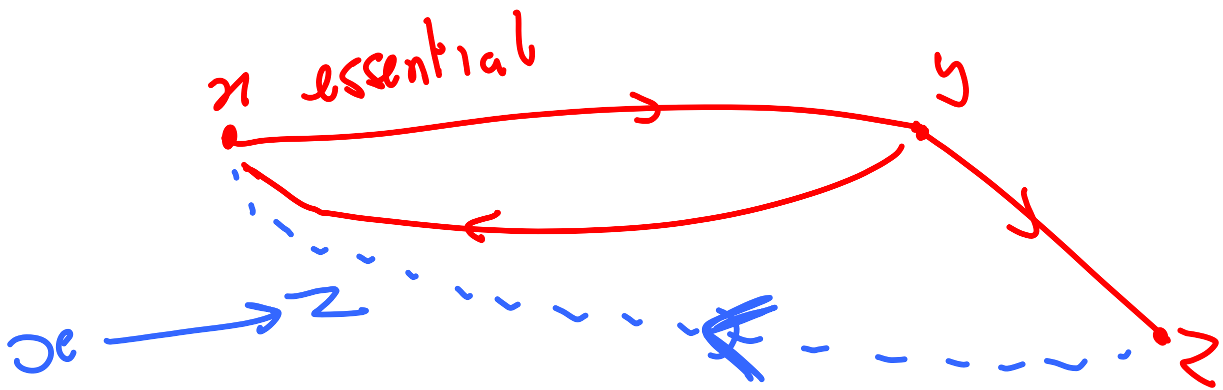


Communicating Class ≡ strong component
Defined by equivalence relation $x \sim y$: $x \to y$ & $y \to x$



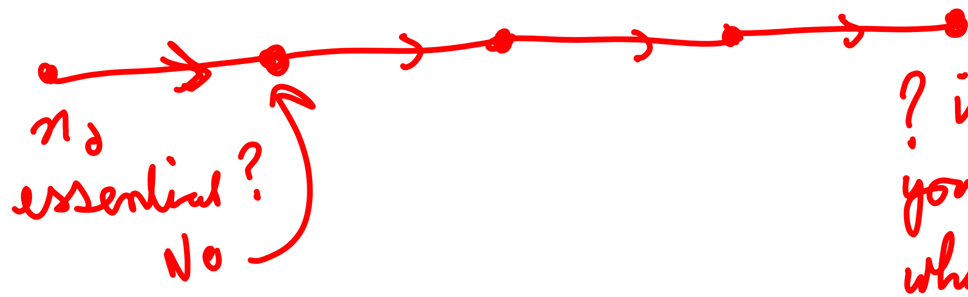
Communicating class

All essential or all inessential



L 1.24

Every chain has at least one essential class. (state)



? is there a state you can reach which cannot reach you?

P 1.25

If π is stationary then $\pi(x) = 0$ for all inessential states.

Pf

$\mathcal{C}$ = essential class.

$$\pi P(\mathcal{C}) = \sum_{z \in \mathcal{C}} (\pi P)(z)$$

$$= \sum_{z \in \mathcal{C}} \left(\sum_{y \in \mathcal{C}} \pi(y) P(y,z) + \sum_{y \notin \mathcal{C}} \pi(y) P(y,z) \right)$$

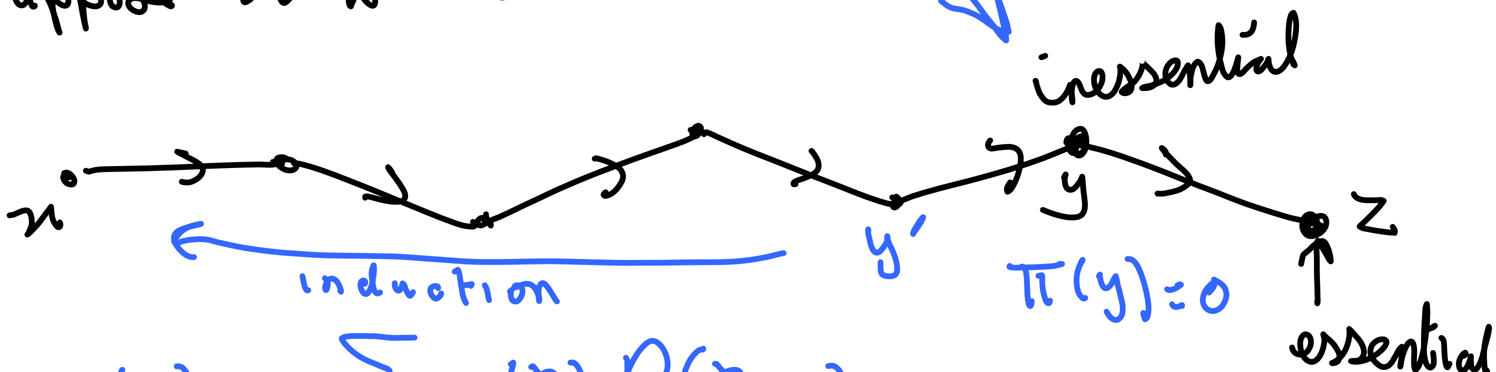
$$= \sum_{y \in \mathcal{C}} \pi(y) \overbrace{\sum_{z \in \mathcal{C}} P(y,z)}^{=1} + \sum_{z \in \mathcal{C}} \sum_{y \notin \mathcal{C}} \pi(y) P(y,z)$$

$$\overbrace{\pi P(\mathcal{C}) = \pi(\mathcal{C})}^{\text{same}} + \underbrace{\sum_{z \in \mathcal{C}} \sum_{y \notin \mathcal{C}} \pi(y) P(y,z)}_{=\,0}$$

$$\pi P = \pi$$

$$\pi(y) P(y,z) = 0 \qquad z \in \mathcal{C},\ y \notin \mathcal{C}$$

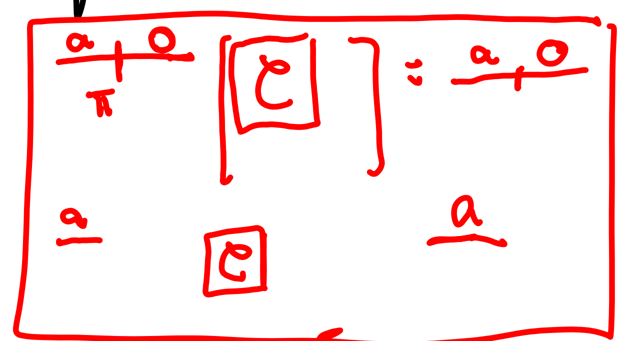
Suppose x is inessential.



$$0 = \pi(y) = \sum_{\xi \in \Omega} \pi(\xi) P(\xi, y) \geq \underset{\substack{\| \\ 0}}{\pi(y')} P(y', y)$$

P1.26

The stationary distribution π is unique iff $\exists$ a unique essential class.



Pf

Unique class $\mathcal{C}$.

$P_{|\mathcal{C}}$ is irreducible $\Rightarrow$ $\pi_{|\mathcal{C}}$ is unique.

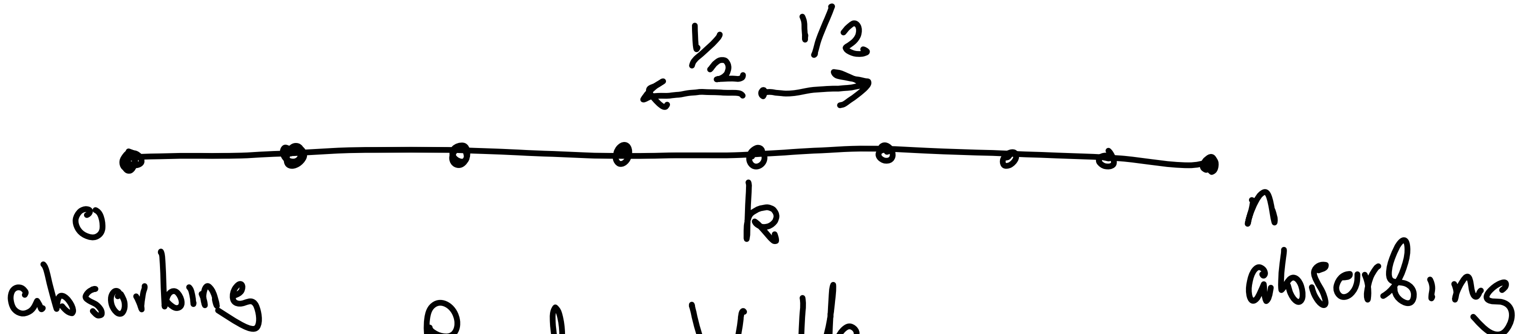
Markov Chain Restricted to $\mathcal{C}$

Suppose π is stationary, then $\pi(y) = 0$, $y \notin \mathcal{C}$ and π is stationary for $P_{|\mathcal{C}}$ must equal $\pi_{|\mathcal{C}}$

Two essential states C_1, C_2 give rise to 2 stationary distributions

Chapter 2

1) Gambler's Ruin



Random Walk

X_t = fortune at time t

τ = absorption time.

$$p_k = Pr_k(X_\tau = n) = k/n$$

$$E_k(\tau) = k(n-k)$$

Pf

(i) $$P_k = \frac{1}{2} P_{k-1} + \frac{1}{2} P_{k+1} \qquad P_0 = 0 \,;\ P_n = 1$$

$$Pr_k(X_\tau = n) = Pr_k(X_\tau = n \mid X_1 = k-1)\, Pr(X_1 = k-1)$$

$X_0 = k$

$$+ \underbrace{Pr_k(X_\tau = n \mid X_1 = k+1)}_{Pr_{k+1}(X_\tau = n)}\, \underbrace{Pr(X_1 = k+1)}_{1/2}$$

(ii) $$f_k = E_k(\tau) = \frac{1}{2}\left(1 + f_{k-1}\right) + \frac{1}{2}\left(1 + f_{k+1}\right)$$

$0 < k < n$

$f_0 = 0$

$f_n = 0$

$k(n-k)$ is the solution.

Coupon Collecting

As a Markov Chain

X_t = # of distinct coupons at time t

$$P(k, k+1) = \frac{n-k}{n}$$

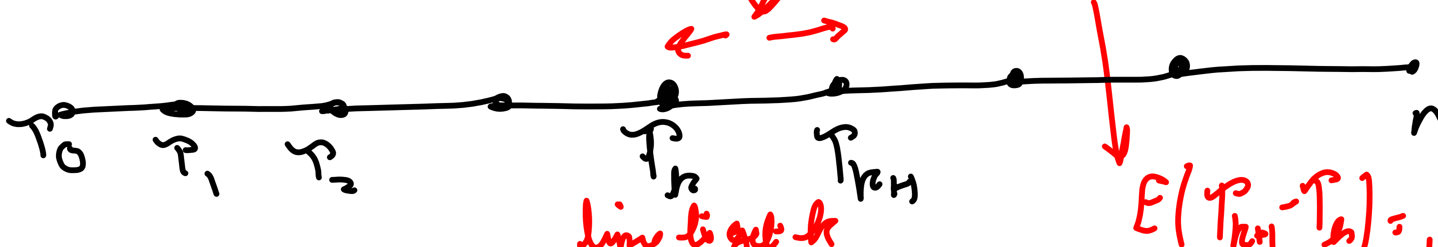
$$P(k, k) = k/n$$

flip coins with $\Pr(H) = 1 - \frac{k}{n}$

T = time to reach n

$$E(T) = n \sum_{k=1}^{n} \frac{1}{k}$$

geometric with $P(\text{success}) = 1 - \frac{k}{n}$



$$E(T_{k+1} - T_k) = \frac{1}{1 - \left(1 - \frac{k}{n}\right)} = \frac{n}{k}$$

2.3

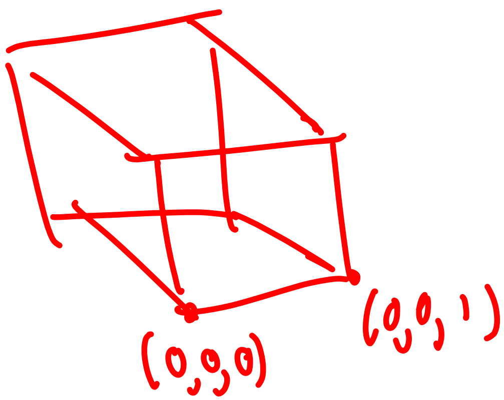
Hypercube and Ehrenfest Urn model

$Q_n = \{0,1\}^n$; $\quad x, y \in Q_n$, $h(x,y) = \# i : x_i \neq y_i$

Hamming Distance

Graph: Vertices Q_n

$x \sim y \Leftrightarrow h(x,y) = 1$



Random Walk on Cube

Lazy Random Walk

$P[x,x] = \frac{1}{2}$

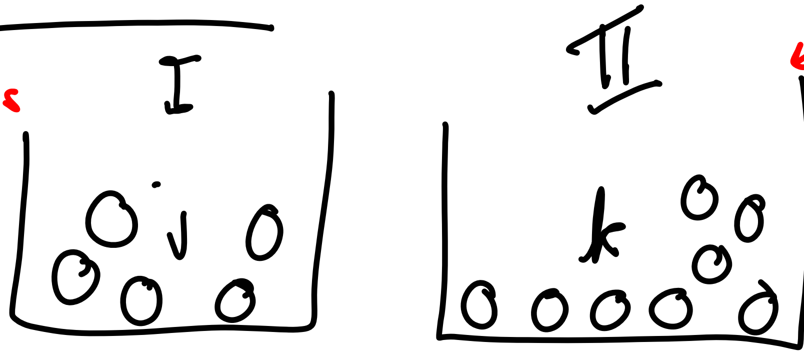
$P[x,y] = \frac{1}{2n} \quad h(x,y) = 1$

Walk: $Z_0, Z_1, \dots Z_n, \dots$

Ehrenfest Urn

1's in Z_t

0's in Z_t



$j + k = n$

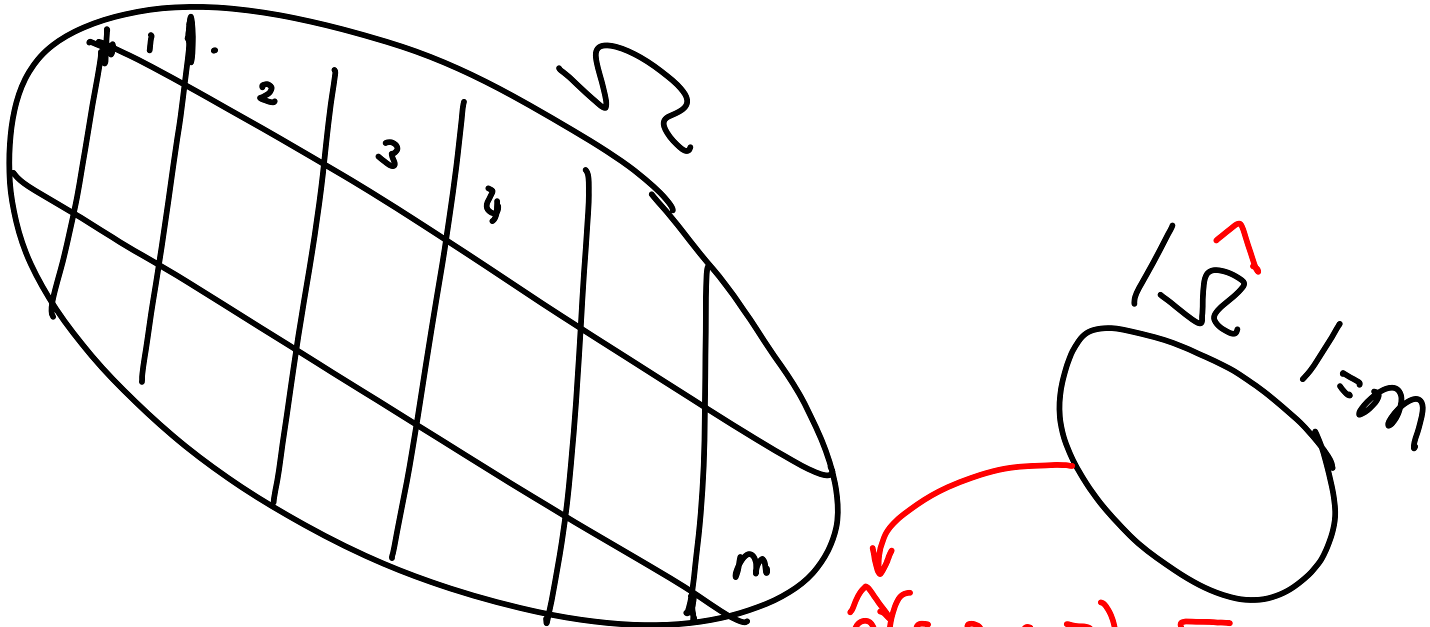
Pick random ball and move to opposite urn.

X_t = # balls in urn I.

$$P[j, k] = \begin{cases} 1 - j/n & k = j+1 \\ j/n & k = j-1 \\ 0 & \text{otherwise} \end{cases}$$

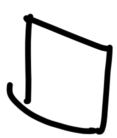
$X_t = h(Z_t, 0)$

Um $\equiv$ projection of RW on Q_n.



$$\hat{p}([x],[y]) = \sum_{z \in [y]} p(x,z)$$

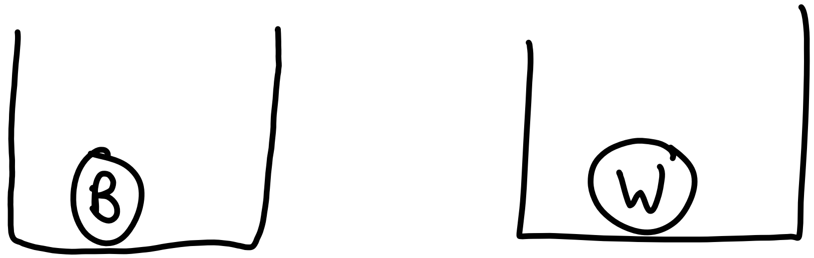
must be independent of $\xi \in [x]$



$\leftarrow$ {states with same number of 1's}

Polya Urn

Initially



Choose a random ball

Add a new ball of that color.

$B_k = \#$ of black balls at time k

Thm: B_k is uniform over $\{1, 2, \dots, k+1\}$

Proof by induction

$$\Pr(B_k = j) = \overbrace{\Pr(B_k = j \mid B_{k-1} = j)}^{1 - j/(k+1)} \overbrace{\Pr(B_{k-1} = j)}^{\text{induction: } 1/k}$$

$$+ \underbrace{\Pr(B_k = j \mid B_{k-1} = j-1)}_{(j-1)/(k+1)} \underbrace{\Pr(B_{k-1} = j-1)}_{1/k}$$

$$= \left(1 - \frac{j}{k+1}\right) \cdot \frac{1}{k} + \frac{j-1}{k+1} \cdot \frac{1}{k}$$

$$= \left(1 - \frac{1}{k+1}\right) \cdot \frac{1}{k}$$

$$= \frac{1}{k+1}$$