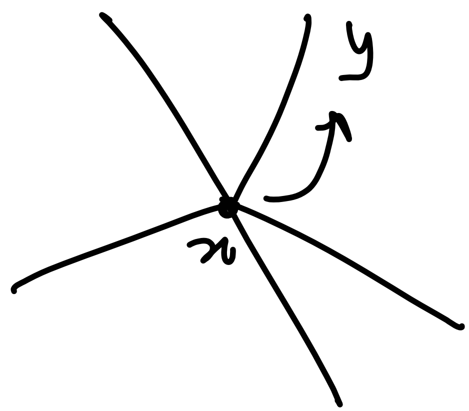
1\13\10

1.4 Random walks on graphs



$G = (V, E)$

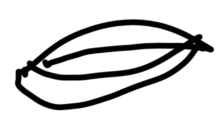
$\Omega = V$

$$\Pr[X_{t+1} = y \mid X_t = x] = \begin{cases} \frac{1}{\deg(x)} & (x,y) \in E \\ 0 & \text{otherwise} \end{cases}$$

Make it more general:

allow (i) loops

(ii) parallel edge



General Reversible chain.

1.5 Stationary Distribution

π is a distribution on Ω : $\sum_{x \in \Omega} \pi_x = 1$

$\pi \geq 0$

π is stationary if $\pi P = P$

or

$$\pi(y) = \sum_{x \in \Omega} \pi(x) P(x, y)$$

Choose x with prob. π and make one move get y with prob π.

π is preserved by taking steps

Example: Random walk on G

$$\pi(x) = \frac{\deg(x)}{2|E|}$$

$$\pi(y) = \sum_{(x,y)\in E} \frac{\deg(x)}{2|E|} \cdot \frac{1}{\deg(x)} = \frac{\deg(y)}{2|E|}$$

$\underline{P}$ $\qquad P\underline{1} = \underline{1} \Rightarrow 1$ is an eigenvalue

stochasticity

$\Downarrow$

$\exists\, u$ s.t. $uP = u$

When does $\exists\, \underline{u} > 0$

Hitting Times

hitting time

$$\tau_x = \min\{t \geq 0 : X_t = x\}$$

depends on initial X_0

$$\tau_x^+ = \min\{t \geq 1 : X_t = x\}$$

first return time

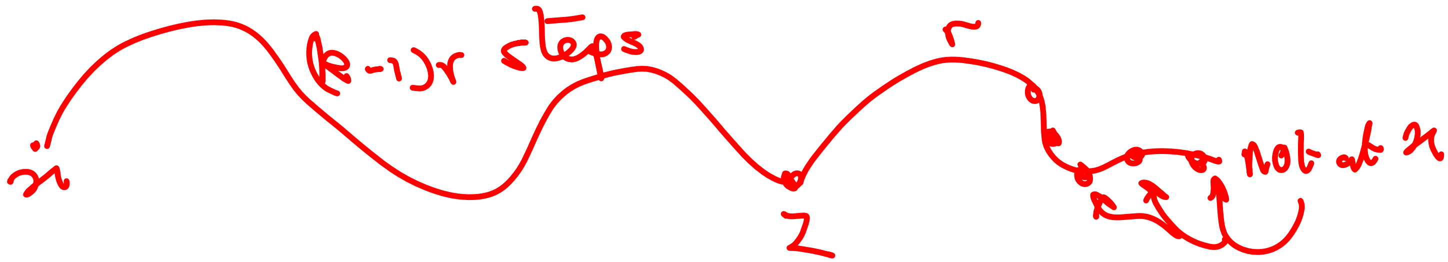
Lemma 1.13 If P is irreducible then

$$E(\tau_x^+) < \infty$$

Proof

$\exists\ \epsilon, r$ s.t. $\forall z, w\ \exists\ j \le r$ s.t. $P^j(x,y) \ge \epsilon$.

$$P_x(\tau_y^+ > kr) \le P_r(\tau_y^+ > (k-1)r)\,(1-\epsilon)$$



$$P_1(T_x^+ > kr) = \sum_{z \neq x} P_1(X_0, X_1, \dots, X_{(k-1)r-1} \neq x,\ X_{(k-1)r} = z,$$
$$X_{(k-1)r+1}, \dots X_{kr} \neq x)$$

start at σ

$$= \sum_z P_\sigma(X_0, X_1, \dots, X_{(k-1)r-1} \neq x,\ X_{(k-1)r} = z) \times \underbrace{P_z(T_x^+ > r)}_{\leq (1-\epsilon)}$$

$\underline{\text{So}}$

$$P_1(T_x^+ > kr) \leq (1-\epsilon)^k$$

$$E(T_x^+) = \sum_{t \geq 1} P_1(T_x^+ \geq t)$$

$E(Z)$

$= p_1 + 2p_2 + 3p_3$

$= p_1 + p_2 + \dots$

$+ p_2 + \dots$

$+ p_3 + \dots$

$$E(\tau_x^+) = \sum_{t \geq 1} P_x(\tau_x^+ \geq t)$$

decreases with t

$$\leq \sum_{k \geq 0} r\, P_x(\tau_x^+ > kr)$$

	k
$1, 2, \ldots r$	1
$r+1, \ldots 2r$	2

$$\leq \sum_{k \geq 0} r(1-\epsilon)^k$$

$$< \infty$$

1.14

P irreducible $\Rightarrow \exists\, \pi > 0$ s.t.

$\pi P = \pi$ [aperiodic, add $P^t \to \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}$ $P^t(x,y) \to \pi_y$

Proof choose $z \in \Omega$

$\tilde{\pi}(y) = E_z$ (no. of visits to y before first return to z) $< \infty$

$\tilde{\pi}(z) = 1$

$\tilde{\pi} > 0$ $= \sum_{t=0}^{\infty} P_z(X_t = y \text{ and } T_z^+ > t)$

show $\tilde{\pi} P = \tilde{\pi}$

$$\sum_{x\in\Omega} \tilde{\pi}(x) P(x,y) = \sum_{x\in\Omega} \sum_{t=0}^{\infty} \underbrace{P_z(X_t = x, \tau_z^+ > t) P(x,y)}_{P_z(X_t = x,\, X_{t+1} = y,\, \tau_z^+ \geq t+1)}$$

(i) $x \neq z$

$X_t = x$

Haven't visited z

in $1, 2, \ldots, t$

(ii) $x = z$

$t = 0$ ✓ $P[X_t = x, \tau_z^+ > 0] = 1$

$t > 0$ " $= 0$

$$= \sum_{t=0}^{\infty} \sum_{x\in\Omega} P_z(X_t = x, X_{t+1} = y, \tau_z^+ \geq t+1)$$

$$= \sum_{t=0}^{\infty} \sum_{x \in \Omega} P_z(X_t = x, X_{t+1} = y, \tau_z^+ \geq t+1)$$

$$= \sum_{t=0}^{\infty} P_z(X_{t+1} = y, \tau_z^+ \geq t+1)$$

$$= \sum_{t=1}^{\infty} P_z(X_t = y, \tau_z^+ \geq t)$$

$$= \tilde{\pi}(y) - P_z(X_0 = y, \tau_z^+ > 0) + \sum_{t=1}^{\infty} P_z(X_t = y, \tau_z^+ = t)$$

(i) $y = z \qquad -1 \quad + 1$

(ii) $y \neq z \qquad -0 \quad + 0$

$$\Rightarrow \quad \tilde{\pi} P = \tilde{\pi}$$

$$\tilde{\pi}(y) = \sum_{t=0}^{\infty} P_z(X_t = y \text{ and } \tau_z^+ > t)$$

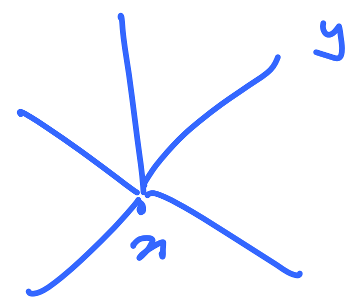
Uniqueness

Harmonic w.r.t. P

$h : \Omega \to \mathbb{R}$

h is **harmonic** at x if

$$h(x) = \sum_{y \in \Omega} P(x,y)\, h(y)$$



$h(x) = E[h(y) : \text{one step}]$

$h = \text{constant}$ is always harmonic

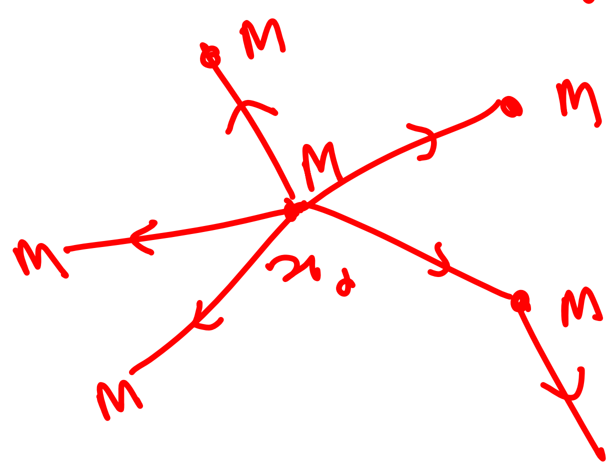
If h is harmonic $\forall x \in \Omega$ then $Ph = h$

L 1.12

P is irreducible and h is harmonic $\forall x \in \Omega$

$\Rightarrow$ h is constant.

$$h(x_0) = M = \max_x h(x)$$



average of nbrs = max

Cor: 1.13 $\quad P$ irreducible $\Rightarrow \pi$ is unique

L1.13 $\Rightarrow$ $P - I$ has rank $|\Omega| - 1$