1\11\2010

Finite Markov Chains

Ω is a finite set.

$X_0, X_1, \ldots, X_t, \ldots$
is a sequence of random variables on Ω.

$$\Pr[X_{t+1} = y \mid X_0, X_1, \ldots, X_t = x]$$
$$= \Pr[X_{t+1} = y \mid X_t = x]$$

Markov Property

$= P(x, y)$

Prob. of going from state x to state y in one step.

$$\sum_y P(x, y) = 1$$

P is (i) non-negative

(ii) Stochastic

Example 1


Pad 1 Pad 2

States are $\Omega = \{1, 2\}$

$$P = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \end{array}$$

On pad 1 go to pad 2 with prob. p, else stay put.

$$P(x,y) = \Pr[x \longrightarrow y \text{ in one step}]$$

$$P^t(x,y) = \Pr[x \longrightarrow y \text{ in } t \text{ steps}]$$

Induction on t.

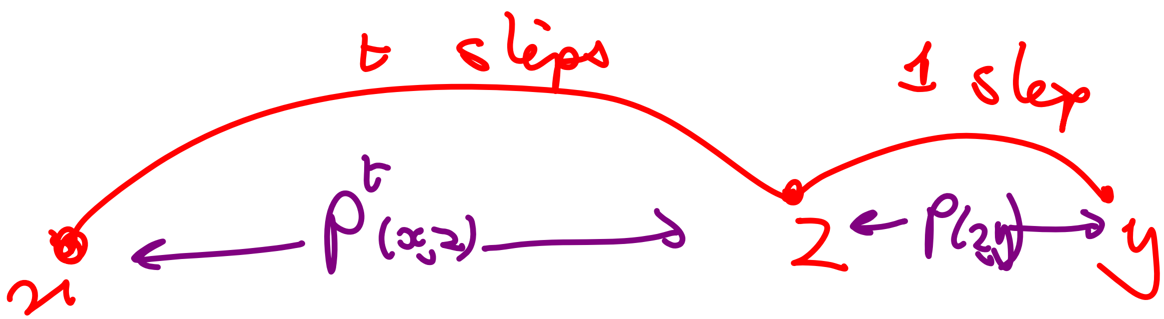
True for $t = 1$.

Assume true for some $t \geq 1$.

$$P^{t+1}(x,y) = \sum_{z \in \Omega} P^t(x,z) P(z,y)$$

$$P^{t+1}(x,y) = \sum_{z \in \Omega} P^{t}(x,z)\,P(z,y)$$

Going $x \longrightarrow y$ in $t+1$ steps



Different z define disjoint events

Start in state x with probability $\mu(x)$

Probability in y after t steps =

$$\sum_{x} \mu(x) P^{t}(x,y) = (\mu P^{t})_{y}$$

μP^t $\quad\quad$ $t \to \infty$

In many chains of interest

Pr(being here, starting at 1)

$$
\begin{array}{c} 1 \\ \\ \\ n \end{array}
\begin{bmatrix} * & * & * & * & * & * \\ & & & & & \\ & & & & & \\ * & * & * & \cdots & & * \end{bmatrix}
\longleftarrow P^t
$$

Pr(being here, starting at n

$$
\begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \\ \pi_1 & \pi_2 & \cdots & \pi_n \\ \vdots & & & \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{bmatrix}
$$

all rows look same

The probabilities $\pi_1, \dots \pi_n$ are called the <u>steady state distribution</u>

Irreducibility and Aperiodicity

P is irreducible if $\forall x, y$ there exists a $t = t(x,y)$ such at $P^t(x,y) > 0$

Ω : directed graph (Ω, A)

Irreducible $\equiv$ strongly connected

$\exists$ directed path $x \longrightarrow y$ in D, $\forall x, y$

$(x,y) \in A$ $\leftrightarrow$ $P(x,y) > 0$

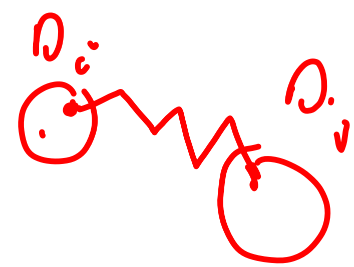
$\underline{D}$: define a relation $\sim$

$$a \sim b \equiv \exists \text{ path } a \to b \text{ \& path } b \to a$$

$\sim$ is an equivalence relation

Equivalence classes are called "strong components"

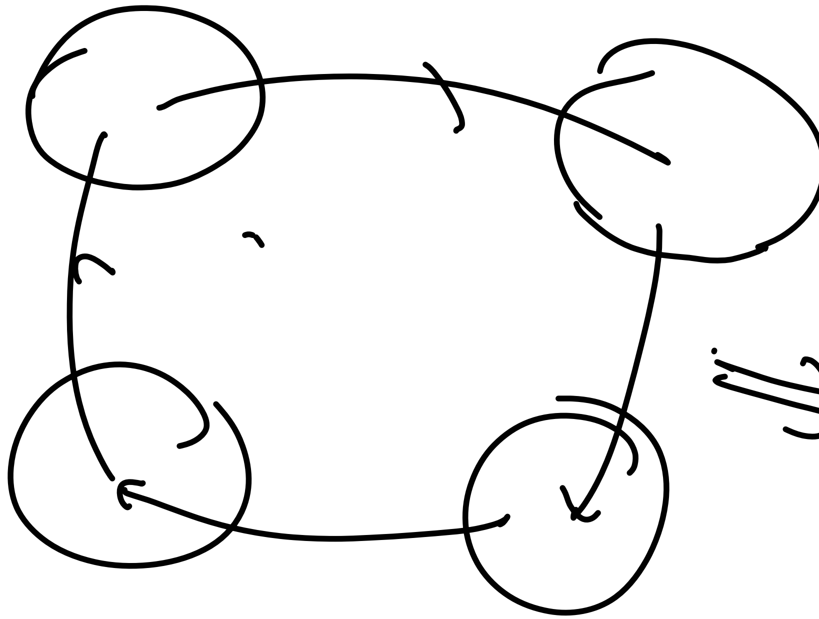
$D_1, D_2, \ldots, D_k$



Define $\Gamma = (\{1, 2, \ldots, k\}, B)$

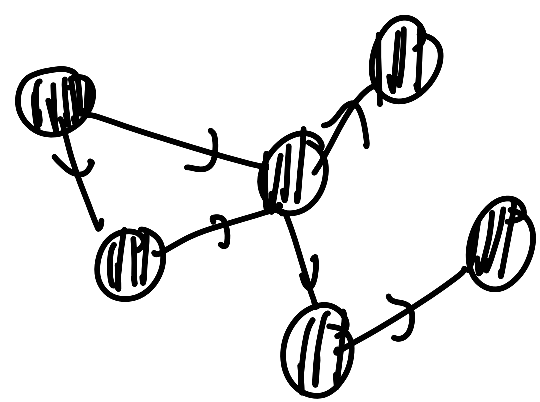
$(i,j) \in B$ if $\exists$ $x \in D_i$, $y \in D_j$ and path $x \to y$ in D

Γ is acyclic:

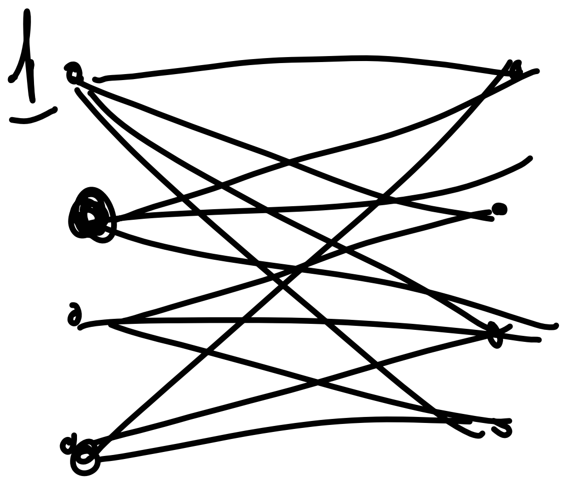


$\implies$ all in same component.

Γ is a DAG:



Periodicity



Start at 1, at even times, on left
at odd times, on right

no steady state.

Fix this: by putting a loop at each vertex.

LAZY CHAIN

$$P \to \frac{I+P}{2}$$

$$P(x,x) \geq \frac{1}{2} \quad \forall x$$

$$\mathcal{T}(x) = \{ t : P^t(x,x) > 0 \}$$

Period of x = gcd $\mathcal{T}(x)$

<u>Lemma</u>

If P is irreducible then

$$\gcd \mathcal{T}(x) = \gcd \mathcal{T}(y), \quad \forall x, y$$

= period of chain

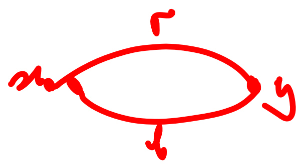
Proof

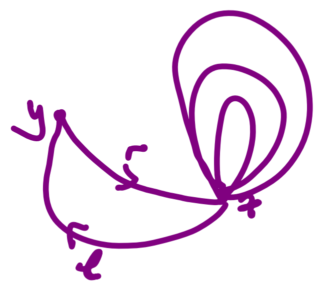
Fix two states x, y.

$\exists\, r, l$ such that

$$P^r(x,y) > 0 \quad \& \quad P^l(y,x) > 0$$

$$m = r + l \in T(x) \cap T(y)$$





$$T(x) \subseteq T(y) - m$$

$$\Downarrow$$

$$\{ x_0, x_1, \dots \}$$

m

gcd $T(y)$ divides everything in $T(x)$

$$x_i = y_i - \underset{\substack{\| \\ y_0}}{x_0}$$

$\square$

Proposition 1.7

P is aperiodic and irreducible $\Rightarrow \exists\, r$ such that $P^r(x,y) > 0 \quad \forall x,y$

Proof

Fact: If $a_1, a_2, \dots, a_k$ are positive integers with gcd 1 then $\exists\, n_0$ such that if $n \geq n_0$ then $n = \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_k a_k$ where $\alpha_1, \alpha_2, \dots, \alpha_k$ are non-negative integers

(Frobenius number)

Another way of phrasing:

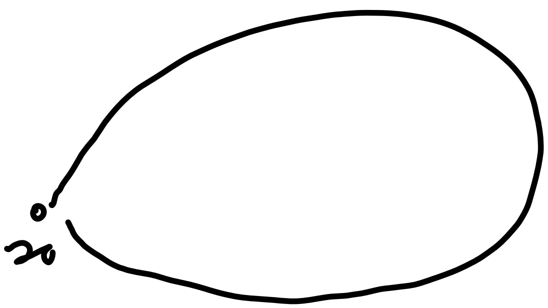
$A \subseteq \{1, 2, \ldots \}$

(i) $\gcd A = 1$

(ii) A is closed under addition

$\Longrightarrow \exists\, n_0$ s.t. $n \geq n_0 \Rightarrow n \in A$.

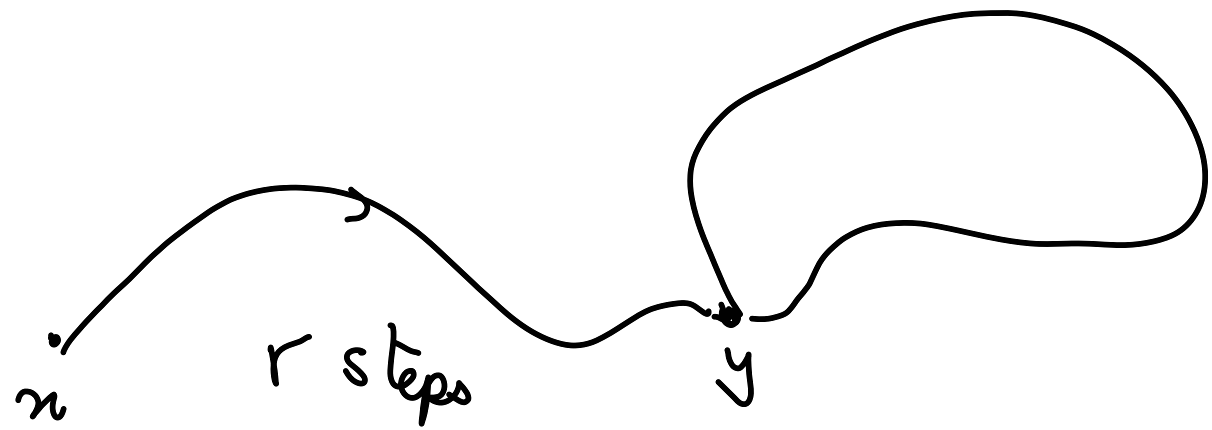
$T(x)$'s fit this claim.



x_0

$P^n(x,x) > 0$

$n \geq n_0 = \max(n_1, n_2, \ldots, n_{|\Omega|})$



$$P^t(x,y) > 0 \quad \text{for} \quad t \geq r + n_0$$

C.S. example of Markov Chains

$G = (V, E)$ of maximum degree Δ

Suppose $k > \Delta$

Then $\exists$ a k-coloring of G.

[Give each vertex a color from $1, 2, \ldots, k$ such that adjacent vertices get different color.]

Suppose I want to count # ways of k-coloring G.

(i) Difficult to do exactly

(ii) Approximately?

$\Omega = \{ k\text{-colorings of } G \}$

$\Omega \neq \emptyset$ if $k > \Delta$

Suppose I could choose a random coloring. — (Markov chain will do this)

Choose large number

v, v not adjacent to w

w

Can estimate ratio $\dfrac{\#\text{colorings } c(v) = c(w)}{\#\text{colorings } c(v) \neq c(w)}$

Generating a random member of Ω

Define Markov chain on Ω.

Given a coloring $\omega \in \Omega$

(i) chose random vertex

(ii) randomly recolor.

(i) $k \geq \Delta + 2 \Longrightarrow$ (i) chain is "ergodic"

(ii) uniform steady state

Run chain for a "long time" and take coloring

$k > 2\Delta$ $O(n \log n)$ is enough