Department of Mathematical Sciences

21–241 **Matrix Algebra** **Fall 2002**

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Test No. 3

Name:______________________________

Section:____________________________

problem	points	scores
1	25	
2	25	
3	25	
4	25	
total	100	

1. (25 pts)

(a) Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}.

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \quad 4\ pts \\ A^{-1}A\mathbf{x} &= \lambda A^{-1}\mathbf{x} \quad 4\ pts \\ A^{-1}\mathbf{x} &= \lambda^{-1}\mathbf{x} \quad 4\ pts \end{aligned}$$

(b) Show that if A^2 is the zero matrix then the only eigenvalue of A is zero.

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \quad 3\ pts \\ A^2\mathbf{x} &= \lambda A\mathbf{x} \quad 3\ pts \\ &= \lambda^2\mathbf{x} \quad 2\ pts \end{aligned}$$

So

$$0 = \lambda^2\mathbf{x} \quad 2\ pts$$

and then $\lambda = 0$, since $\mathbf{x} \neq 0$ 3 pts.

2. (25 points) Let

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

Diagonalise A.

$$det\ A - \lambda I = (\lambda - 5)(\lambda + 2) \qquad 5\ pts$$

Eigenvalues are 5 and -2 – 4 pts.

Eigenvectors are: 5: [1 1] (4 pts) and -2: [3 $-$ 4] (4 pts).

$$P = \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix} \qquad 4\ pts$$

$P^{-1}AP$ is diagonal. 4 pts.

3. (25 points) An $n \times n$ matrix is *orthonormal* if its columns are an orthonormal set.

(a) Show that the eigenvalues of an $n \times n$ orthonormal matrix A are ± 1.

(Hint: consider the length of vector $A\mathbf{x}$ when $\mathbf{x}$ is an eigenvector of A.)

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \qquad 3\ pts \\ \mathbf{x}^T A^T A\mathbf{x} &= \lambda^2 ||\mathbf{x}||^2 \qquad 3\ pts \\ \mathbf{x}^T\mathbf{x} &= \lambda^2 ||\mathbf{x}||^2 \qquad 3\ pts \end{aligned}$$

So $\lambda^2 = 1$ since $\mathbf{x} \neq 0$ 3 pts.

(b) Let U, V be $n \times n$ orthonormal matrices Show that UV is also orthonormal.

$$\begin{aligned} (UV)^T UV &= V^T U^T UV \qquad 3\ pts \\ &= V^T V \qquad 3\ pts \\ &= I \qquad 3\ pts \end{aligned}$$

This implies that UV is orthonormal 4pts.

4. (25 points)

Let $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

Write $\mathbf{y}$ as the sum of a vector in Span$\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

$$\begin{aligned} \mathbf{y} &= \left(\mathbf{y} - \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) + \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}. \\ &= \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix} + \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}. \end{aligned}$$

10 pts for correct method and rest for accuracy. Take only a few points off for numerical mistakes.