Department of Mathematical Sciences

21–241 **Matrix Algebra** **Fall 2002**
Professor A. Frieze and Professor S. Ta'asan

Test No. 2

Name:______________________________
Section:____________________________

problem	points	scores
1	25	
2	25	
3	25	
4	25	
total	100	

1. (25 pts)

(a) Give the definition of an invertible matrix.

Answer An $n \times n$ matrix A is invertible if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. (5pts) Anything similar will do.
-1pt for not saying $n \times n$ or square.

(b) If A, B, C are n by n invertible matrices, Does the equation

$$C^{-1}(A + X)B^{-1} = I_n$$

have a solution, X? If so find it.

$$\begin{aligned} (A + X)B^{-1} &= C & 6pts \\ A + X &= CB & 6pts \\ X &= CB - A & 6pts \end{aligned}$$

2 extra pts for perfect answer.
-2 if CB is replaced byBC.

2. (25 points) Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -4 & 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 7 \\ 0 \\ 3 \end{bmatrix}$$

Solve the equation $A\mathbf{x} = \mathbf{b}$.

Let $A = LU$ here.
First solve $Ly = b$ – 2pts

$$\begin{aligned} y_1 &= 1 && 2pts \\ y_2 &= 5 && 2pts \\ y_3 &= 1 && 2pts \\ y_4 &= -3 && 2pts \end{aligned}$$

Then solve $Ux = y$ – 2pts

$$\begin{aligned} x_1 &= -2 && 2pts \\ x_2 &= -1 && 2pts \\ x_3 &= 2 && 2pts \\ x_4 &= -3 && 2pts \end{aligned}$$

5 extra pts for a perfect answer.

Lose at most 5 pts for numerical errors, as long as they show that they are solving $Ly = b$ and then $Ux = y$ by substitution.

A reduction to solving $Ly = b$ and $Ux = y$, but then using the general method i.e reducing L for example to echelon form will be graded out of 20pts.

3. (25 points)

(a) Define the column space and the null space of a matrix A.

col A is the subspace spanned by the columns of A – 3pts.
nul A is the set of $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{0}$ – 3pts.

(b) Find a basis for the column space and the null space of the following matrix.

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 3/2 \\ 0 & 0 & 1 & 5/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{b}_1 = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} \quad \text{is a basis for col } A \text{ – 4pts+4pts.}$$

$$\mathbf{b}_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_4 = \begin{bmatrix} -3/2 \\ 0 \\ -5/4 \\ 1 \end{bmatrix} \quad \text{is a basis for nul } A \text{ – 4pts+4pts.}$$

3 extra pts for a perfect answer.

4. (25 points) Given $\mathbf{u}$ in $\mathbb{R}^n$ with $\mathbf{u}^T\mathbf{u} = 1$, let $P = \mathbf{u}\mathbf{u}^T$, and $Q = I - 2P$. Prove

(a) $P^2 = P$,

$$P^2 = \mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T = \mathbf{u}(\mathbf{u}^T\mathbf{u})\mathbf{u}^T = \mathbf{u}(1)\mathbf{u}^T = P. \qquad \text{8 pts}$$

(b) $P^T = P$,

$$P^T = (\mathbf{u}\mathbf{u}^T)^T = (\mathbf{u}^T)^T\mathbf{u}^T = \mathbf{u}\mathbf{u}^T = P \qquad \text{6 pts}$$

(c) $Q^2 = I$.

$$\begin{aligned} Q^2 = (I-2P)(I-2P) = (I-2P)I - (I-2P)2P &= I - 2P - 2P + 4P^2 \\ &= I - 2P - 2P + 4P = I \qquad \text{10 pts} \end{aligned}$$

1pt extra for a perfect answer.