Coupling of Scale-Free and Classical Random Graphs

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April 18, 2007

Introduction

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Just how robust are scale free graphs

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- should the adversary have infinite time?

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How many vertices do we need to remove so that the graph breaks up into small pieces?

The Model

Consider the Preferential Attachment Model.
Assume m is fixed, $G[k] \sim G_m^{(k)}$. We create $G[k+1]$ by adding a new vertex $k+1$ with m edges $(k+1, t_1), \ldots, (k+1, t_m)$ where

$$\Pr[t_i = j] = \frac{d_{G[k],i}(j)}{2mk + 2i - 1}$$

Main Result

Theorem 1

There exist $c > 0, m_0$ s.t. for $m \geq m_0$, **whp** $G_m^{(n)}$ the following holds

- (a) Every induced subgraph of size $10n\frac{\log(m)}{m}$ contains a component of size at least $2n\frac{\log m}{m}$
- (b) The graph contains an independent set of size $cn\frac{\log(m)}{m}$

Proof Technique

Proof by coupling

We create two graphs (G_1, G_2) s.t. $G_1 \sim G_m^{(n)}$, $G_2 \sim G(n,p)$.

But G_1 and G_2 are not independent but heavily correlated.

What is coupling?

Let $X \sim D_X$, $Y \sim D_Y$. A coupling is a joint distribution $D_{X,Y}$ s.t. the marginals are correct, i.e. $D_{X,\cdot} = D_X$ and $D_{\cdot,Y} = D_Y$.

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Chernoff Bounds

Let $X \sim Bin(n,p)$ then

$$\Pr[X > (\delta+1)np] < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{np}$$

$$\Pr[X < (1-\delta)np] < \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{np}$$

The Coupling

Theorem 2 Fix $\eta < 1/2$, there exist constants $A, c > 0$ s.t. for fixed m we can construct $G_1 \sim G_m^{(n)}$, $G_2 \sim G(n, \eta m/n)$ s.t. **whp** $e(G_2 \setminus G_1) \leq Ae^{-cm}n$

The Coupling
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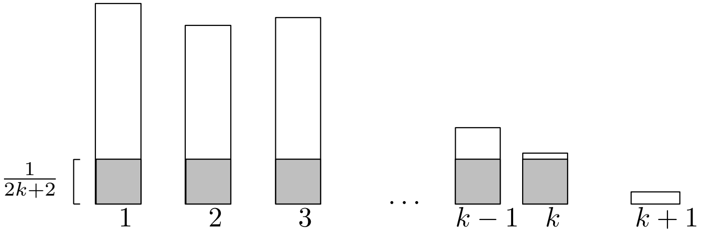
- Given $G[k]$
  - generate $Y \sim Bin(k,p)$
  - let S be a random subset of $[k]$ of size Y
  - create edges between k and S

We grow G_1, G_2 one node at a time. Start with $G_1[k_0] \sim G_m^{(n)}$, $G_2[k_0] \sim G(k_0, \eta m/n)$, where k_0 grows slowly with n.
Let $t_1, \ldots, t_m$ be the vertices picked when growing $G_1[k]$. For $j \neq k+1$ we have

$$\Pr(t_i = j) = \frac{d_{G[k],i}(j)}{2mk + 2i - 1} \geq \frac{m}{2km + 2m} = \frac{1}{2k+2}$$



Construct $s_1, \ldots, s_m \in [k] \cup \varnothing$, s.t. s_i's are independent $\Pr(s_i = j) = \frac{1}{2k+2}$ for $j \neq \varnothing$ and $s_i = j$ implies $t_i = j$.

Let $X = |\{i|s_i \neq \varnothing\}|$, then $X \sim Bin(m, \frac{k}{2k+2})$.
Let $Y \sim Bin(k, \eta m/n)$, then

$$E[X] = \frac{mk}{2k+2} \geq (1+\epsilon)\eta\frac{mk}{n} = (1+\epsilon)E[Y]$$

So $\Pr[X < Y] \leq Ae^{-c'm}$ for some constants $A, c' > 0$.

All the s_i's are distinct w.p $1 - O(\frac{m^2}{k})$, given this and X the set $S_1 = \{s_i | s_i \neq \varnothing\}$ is a random subset of $[k]$ of size X.

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Otherwise we pick S_2 a random subset of size Y from $[k]$, this happens w.p. $Ae^{-c'm} + o(1)$.

The Coupling

Proof

Set $D_{k+1} = \#$of edges added to G_2 and not G_1.
$\Pr[D_{k+1} > 0] = Ae^{-c'm} + o(1)$, and $D_{k+1} \leq Y$ so
$E[D_{k+1}] \leq Ame^{-c'm}$

The number of edges in $G_2 \setminus G_2$ is at most

$$\binom{k_0}{2} + \sum_{k=k_0+1}^{n} D_k$$

Assuming $k_0 = n^{1/4}$ we see that the sum has expected value at most $Ane^{-c'm}$ and is concentrated around its mean.

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Proof of Theorem 1(a)

Let G_1, G_2 be as described in Theorem 2. Suppose G_1 has an induced subgraph on a set V, with $|V| = 10n^{\frac{\log(m)}{m}}$ where every component has size less than $2n^{\frac{\log(m)}{m}}$. Then we can partition V into two set V_1, V_2 s.t. $|V_1|, |V_2| \geq 4n^{\frac{\log(m)}{m}}$ and G_1 has no V_1-V_2 edges.

Proof of Theorem 1(a)

Let $x = Ae^{-c'm}n$, if the coupling works then G_2 has at most x, V_1-V_2 edges.
The number of V_1-V_2 edges in G_2 is a binomial with mean

$$\mu = |V_1||V_2|\eta m/n \geq 24\eta(\log(m))^2 n/m$$

Pick m large enough s.t. $x < \mu/100$

Proof of Theorem 1(a)

By Chernoff bounds the probability that we have fewer than x edges crossing V_1, V_2 is at most $e^{-11\mu/12}$. Now

$$\Pr[G_2 \text{ has at most } x \quad V_1 - V_2 \text{ edges}] \leq \binom{n}{|V|} 2^{|V|} e^{-22\eta(\log(m))^2 n/m}$$

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- We create G_1, G_2 simultaneously s.t.
- a bad event in G_1 implies w.h.p another bad event in G_2
- which is easy to show is rare in G_2.
- But we are working in the joint probability space so this implies that the bad event is rare in G_1.

Another Coupling

Theorem 3
Let $\epsilon > 0$ be given, there is a constant C s.t. for fixed m we can couple G_1, G_2 s.t. $G_1 \sim G_m^{(n)}$, $G_2 \sim G(n, Cm/n)$ s.t. **whp** G_2 contains $G_1 \setminus V$ for a set of vertices V s.t. $|\{i \in V | i \geq \epsilon n\}| \leq \epsilon n/m$.

We start with $G_1[k_0] \sim G_m^{(k_0)}$ and $G_2[k_0] \sim G(k_0, Cm/n)$ independent, and here $k_0 = \epsilon n$.
A vertex is *bad at time k* if it has degree at leas Am in $G_1[k]$.

We need the following result from Bollobás et al.: Vertex i has degree $d+m$ in $G_m^{(n)}$ with probability

$$o(n^{-1}) + (1+o(1))\binom{d+m-1}{m-1}(\frac{i}{n})^{m/2}(1-\frac{i}{n})^d$$

i.e. for $i \geq \epsilon n$ it is exponentially small in d for large d
If we choose A large enough the expected number of bad vertices is at most $e^{-Bm}n$ so **whp** there are at most $\epsilon n/m$ bad vertices.

Let $t_1, \ldots, t_m$ be defined as in the previous proof. Let V_k be the set of vertices j, s.t. $k_0 \leq j \leq k$ that are good. Now for $j \in V_k$

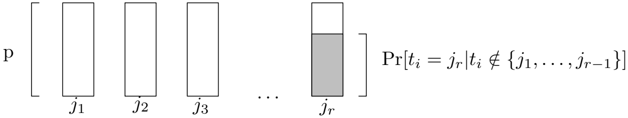
$$\Pr[t_i = j] \leq (Am + m)/(2mk) \leq A/k \leq A\epsilon^{-1}/n$$

and

$$\Pr[t_i \notin V_k] \geq k_0 m/(2mn) = \epsilon/2$$

so $\Pr[t_i = j_r | t_i \notin \{j_1, \ldots, j_{r-1}\}] \leq 2A\epsilon^{-2}/n = p$

Let T_i be a random subset of V_k s.t. every vertex is picked independently with probability p. Then we can couple T_i and t_i s.t. $t_i \in T_i$ if t_i is good.



Let S_1 be the set of good vertices adjacent to $k+1$ in G_1. Let $S_2 = T_1 \cup T_2 \cup \dots T_m$. Then S_2 is a random subset of V_k where every vertex is picked w.p. $1-(1-p)^m \leq mp \leq Cm/n$ for some large C.
This completes the proof since we've constructed S_1 s.t. $S_1 \subseteq S_2$.

For a constant $a > 1$ we know that $G_{n,\frac{a}{n}}$ has an independent set of size $\approx \frac{\log(a)}{a} n$ when $a \to \infty$.
Now couple G_1 and G_2 as in Theorem 3, with $\epsilon = \frac{1}{2}$ and assume $C > 2$. Look at the subgraph of G_2 of vertices that come in after $\frac{n}{2}$.
It has distribution $G_{\frac{n}{2},\frac{Cm}{n}} \sim G_{\frac{n}{2},\frac{Cm/2}{n/2}}$ so it has an independent set of size $\approx n\frac{\log(m)}{Cm}$.
Removing the bad vertices from this set we see that this set is also independent in G_1. So G_1 has an independent set of size $n\frac{\log(m)}{Cm} - \frac{n}{2m}$. By picking m large enough we get the result.

Thank You

The End

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Questions?