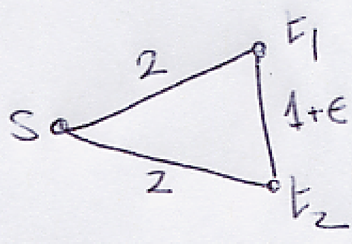
(X)

$$
\begin{aligned}
c(S') &= y_1 + y_2 + y_3 \\
&\leq y_1 + y_2 + \tfrac{4}{3} y_3 \\
&= \tfrac{1}{9}(y_1 + y_2) + \tfrac{8}{9}\left(y_1 + y_2 + \tfrac{3}{2} y_3\right) \\
&\leq \tfrac{4}{9}(x_1 + x_2) + \tfrac{8}{9}\left(x_1 + x_2 + \tfrac{3}{2} x_3\right) \\
&= \tfrac{4}{3}(x_1 + x_2 + x_3) \\
&= \tfrac{4}{3}\, c(S).
\end{aligned}
$$

<u>Example</u>



OPT $= 3 + \epsilon$

NASH $= 4$

Price of stability ....

Digraph D $\qquad c_e$ = cost of edge e

k user's

User i wants to ~~construct a~~ use a path

S_i from s_i to t_i.

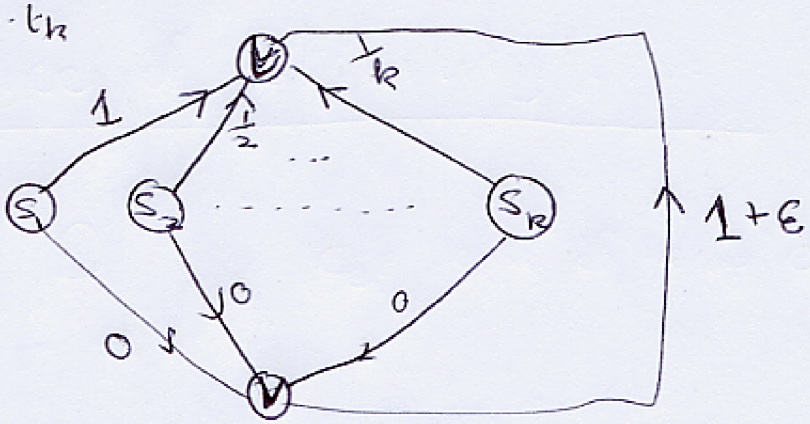
Cost $C_i(S_1, S_2, \dots, S_k) = \sum_{e \in S_i} \frac{c_e}{|\{j : e \in S_j\}|}$

< to user i >

Total cost $= \sum_i C_i = \sum_{e \in S_1 \cup \dots \cup S_k} c_e$.

Question: what is $\underbrace{\min \frac{\text{cost of Nash}}{\text{cost of minimum}}}_{\text{Price of stability}}$

$t = t_1, t_2, \cdots t_k$



Optimal solution: all users $s_i \to v \to t$

Total cost $1+\varepsilon$.

N.E. Go direct: Total cost $= H(k) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$

Thm 2.1

Price of stability $\leq H(k)$

Proof

$f_e(x) = c_e / x$ $\qquad S = (S_1, S_2, \ldots, S_k)$

$x_e = \#$ users of e under

$$\phi(S) = \sum_{e \in E} \sum_{x=1}^{x_e} f_e(x)$$

Claim

$$\left.\begin{array}{l} S_i \to S_i' \\ S \to S' \end{array}\right\} \quad \Phi(S') - \Phi(S) = C_i(S') - C_i(S)$$

Proof

$$\Phi(S') - \Phi(S) = \sum_e c_e \left(H(x_e') - H(x_e)\right)$$

$$= \sum_{e \in S_i' \setminus S_i} \frac{c_e}{x_e + 1} - \sum_{e \in S_i \setminus S_i'} \frac{c_e}{x_e}$$

$$= C_i(S') - C_i(S).$$

$\square$

Let $S^* = (S_1^*, \dots S_k^*)$ minimize cost.

$$OPT = \sum_{e \in S^*} c_e .$$

(0) $\Phi(S^*) \leq \sum_{e \in S^*} c_e H(k) = H(k) \times OPT$

Start with $S^{(0)} = S^*, S^{(1)}, S^{(2)}, \dots$ $S^{(t+1)}$: some player improves cost w.r.t. $S^{(t)}$.

$\dots S \leftarrow$ N.E.

(i) $\Phi(S) \leq \Phi(S^*)$

(ii) $c(S) \leq \Phi(S)$.

$\square$

Thm 2.2

Replace c_e by $c_e(x)$ — $x = \#$ users of e

$c_e(x)$ concave and $\nearrow$.

Price of stability $\leq H(k)$.

Proof

Re-define $\Phi(S) = \sum_{e \in S} \sum_{x=1}^{x_e} \frac{c_e(x)}{x}$

(i) $\Phi(S') - \Phi(S) = c_1(S') - c_1(S)$, as before.

(ii) $\sum_{x=1}^{x_e} \frac{c_e(x)}{x} \leq c_e(x_e) \cdot H(x_e) \leq c_e(x_e) \cdot H(k)$

$\Rightarrow \Phi(S) \leq c(S) \cdot H(k)$.

(iii) $\frac{c_e(x)}{x} \searrow$, c_e is concave.

$c(S) = \sum_{e \in S} c_e(x_e) = \sum_{e \in S} x_e \cdot \frac{c_e(x_e)}{x_e} = \sum_{e \in S} \sum_{x=1}^{x_e} \frac{c_e(x_e)}{x_e} \leq \Phi(S)$

Argue as in Thm 2.1.

$\square$

Delays

Cost of e to player

$$f_e(x) = \frac{c_e(x)}{x} + d_e(x)$$

Share of construction

delay on e with x users

$$\boxed{\Phi(S) = \sum_{e \in S} \sum_{x=1}^{x_e} f_e(x)}$$

Thm 3.1

Suppose

$$c(S) \leq A\,\Phi(S) \quad \forall S$$
$$\Phi(S) \leq B\,c(S)$$

$\Rightarrow$ Price of stability $\leq A \cdot B$

Proof

$$\Phi(S') - \Phi(S) = \sum_{e \in S_i' \setminus S_i} \left\{ \frac{c_e(x_e+1)}{x_e+1} + d_e(x_e+1) \right\}$$

$$- \sum_{e \in S_i \setminus S_i'} \left\{ \frac{c_e(x_e)}{x_e} + d_e(x_e) \right\}$$

$$= c(S') - c(S)$$

$$\underset{opt}{S^*} \longrightarrow \underset{Nash}{S'} \qquad \Phi(S') \leq \Phi(S^*)$$

$$
\begin{aligned}
c(S') &\leq A\,\Phi(S') \\
&\leq A\,\Phi(S^*) \\
&\leq AB\,c(S^*).
\end{aligned}
$$

Corollary 3.2

$c_e(x)$ is concave $\nearrow$ & $d_e(x)$ $\nearrow$, $\forall e$ &

$$x_e\, d_e(x_e) \leq A \sum_{x=1}^{x_e} d_e(x).$$

Price of stability $\leq A \cdot H(k)$

E.g. $d_e(x) = \mathrm{poly}(x)$ of degree $\leq l$ and non-negative coefficient

$A \leq l+1$.

Proof $\quad \Phi(S) = \underset{\text{cost}}{\Phi_1(S)} + \underset{\text{delay}}{\Phi_2(S)}$

$$c(S) = c_1(S) + c_2(S)$$

$\cancel{\Phi_1(S) \leq}$

$$\Phi_1(S) \leqslant c_1(S) \cdot H(k)$$

$$\Phi_2(S) = \sum_e \sum_{x=1}^{x_e} d_e(x)$$

$$\leqslant \sum_e x_e d_e(x)$$

$$= c_2(S).$$

So
$$\Phi(S) \leqslant c(S) \cdot H(k)$$

$$c_1(S) \leqslant \Phi_1(S)$$

$$c_2(S) = \sum_e x_e d_e(x)$$

$$\leqslant A \sum_e \sum_{x=1}^{x_e} d_e(x)$$

$$= A \, \Phi_2(x).$$

Now apply Thm 3.1.

Undirected Case

Thm

Price of stability is $\le \frac{4}{3}$ with 2 players in an undirected graph, each i trying to connect s to t_i, $i = 1, 2$.

Proof

(S_1, S_2) = optimal centralized solution.

$$X_1 = S_1 \setminus S_2 \; ; \quad X_2 = S_2 \setminus S_1 \; ; \quad X_3 = S_1 \cap S_2 .$$

(S_1', S_2') = Nash obtained from (S_1, S_2) by individual improving moves.

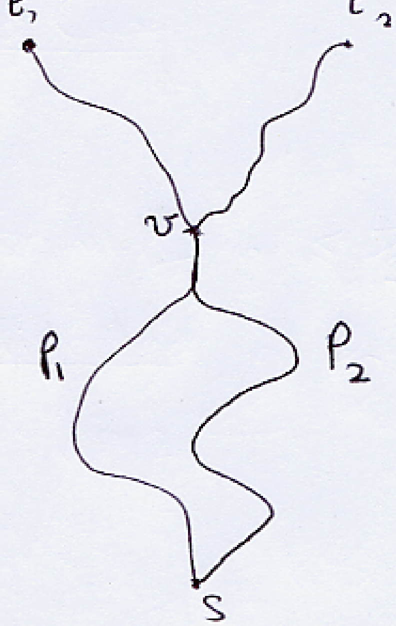
$$X_1^* = S_1' \setminus S_2' \; ; \quad X_2^* = S_2' \setminus S_1' \; ; \quad X_3^* = S_1' \cap S_2'$$

Also $\Phi(S_1', S_2') \le \Phi(S_1, S_2)$

$x_i = c(X_i)$ etc, $y_i = c(Y_i)$

$$y_1 + y_2 + \frac{3}{2} y_3 \le x_1 + x_2 + \frac{3}{2} x_3 \qquad (1)$$

(S_1', S_2')

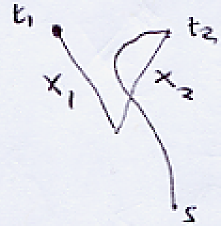


Claim: $P_1 = P_2$

(i) $c(P_1 \setminus P_2) = c(P_2 \setminus P_1)$ else one player will change

(ii) Then if $P_1 \neq P_2$, P_1 will save by using P_2 and sharing costs.

Player 1 could use $X_1 \cup X_2 \cup Y_2 \cup Y_3$ in place of $Y_1 \cup Y_3$



$$x_1 + x_2 + \tfrac{1}{2}y_2 + \tfrac{1}{2}y_3 \geq y_1 + \tfrac{1}{2}y_3$$

also

$$x_1 + x_2 + \tfrac{1}{2}y_1 + \tfrac{1}{2}y_3 \geq y_2 + \tfrac{1}{2}y_3$$

adding

$$\tfrac{1}{2}y_1 + \tfrac{1}{2}y_2 \leq 2x_1 + 2x_2$$