A Class of Games Possessing Pure-Strategy Nash Equilibria

By Robert W. Rosenthal [1])

Abstract: A class of noncooperative games (of interest in certain applications) is described. Each game in the class is shown to possess at least one Nash equilibrium in pure strategies.

1. Description

There are n players ($i = 1,\ldots,n$) and t primary factors ($k = 1,\ldots,t$). The i^{th} player's ($i = 1,\ldots,n$) set of pure strategies contains s_i elements ($r_i = 1,\ldots,s_i$). The r_i^{th} pure strategy may be viewed as the selection of a particular subset of the primary factors. The cost to i of playing the r_i^{th} pure strategy is the sum of the costs of each of the primary factors he selects. The individual factor costs c_k (identical for each player) are functions of x_k, the number of people selecting the k^{th} factor, only. Thus, the cost to player i, if the strategy combination $(r_1,\ldots,r_n)$ is selected, is $\pi_i(r_1,\ldots,r_n) = \sum_{k\in r_i} c_k(x_k(r_1,\ldots,r_n))$. A Nash equilibrium in pure strategies is a pure-strategy combination $(r_1^*,\ldots,r_n^*)$ satisfying

$$\pi_i(r_1^*,\ldots,r_n^*) \le \pi_i(r_1^*,\ldots,r_{i-1}^*,r_i,r_{i+1}^*,\ldots,r_n^*) \quad r_i = 1,\ldots,s_i;\ i = 1,\ldots,n\,.$$

The following are situations in which such a model might be useful.

Example 1:

A network of roads is given. Each of n people must travel (at about the same time) through the network from certain origins to certain destinations. The time it takes to travel on any road is an increasing function of the number of people selecting that road. Each person wishes to travel from his origin to his destination in minimum time. (Another approach to equilibrium traffic flows has received wide attention. See [Charnes and Cooper, 1961], for example. Relationships between that model and this example are discussed in [Rosenthal].)

Example 2:

n firms are engaged in production. Each firm has several alternative production processes available, each of which uses a subset of the primary factors. The use of any primary factor involves a set-up cost but no variable cost. The set-up costs are functions of the number of other firms demanding the same factor. The cost of each process is the sum of the factor costs.

[1]) Robert W. Rosenthal, Department of Industrial Engineering and Management Sciences Northwestern University Evanston, Illinois 60201

2. Pure-Strategy Nash Equilibria

Theorem:

All games of the above class possess at least one pure-strategy Nash equilibrium.

Proof:

Let x_{r_i} be 1 if player i plays strategy r_i, otherwise zero. Consider the problem:

$$(*)\begin{cases} \text{Minimize } \sum_{k=1}^{t} \sum_{y=0}^{x_k} c_k(y) \\ \text{subject to: } \sum_{r_i=1}^{s_i} x_{r_i} = 1 \quad i = 1,\dots,n\,; \\ x_k - \sum_{i=1}^{n} \sum_{r_i \ni k} x_{r_i} = 0 \quad k = 1,\dots,t\,; \\ x_{r_i} = 0 \text{ or } 1 \quad r_i = 1,\dots,s_i\,; \quad i = 1,\dots,n\,. \end{cases}$$

Since solutions to (*) exist, it suffices to show that any solution to (*) gives rise to a pure-strategy equilibrium. Let $\{x_{r_i}^0, x_k^0\}$ solve (*), and suppose the associated strategy combination is not an equilibrium. Then for some j, there is a strategy z_j such that

$$\sum_{\substack{k \in z_j \\ k \notin r_j^0}} c_k(x_k^0 + 1) < \sum_{\substack{k \in r_j^0 \\ k \notin z_j}} c_k(x_k^0)$$

where r_j^0 is the strategy used by j at $\{x_{r_i}^0\}$.

Consider the new values $\{x'_{r_i}, x'_k\}$ associated with player j changing to his z_j^{th} pure strategy (all other players playing the strategies associated with $\{x_{r_i}^0\}$). The objective function evaluated at $\{x'_k\}$ is:

$$\sum_{k=1}^{t} \sum_{y=0}^{x'_k} c_k(y) = \sum_{k=1}^{t} \sum_{y=0}^{x_k^0} c_k(y) + \sum_{\substack{k \in z_j \\ k \notin r_j^0}} c_k(x_k^0 + 1) - \sum_{k \in r_j^0} c_k(x_k^0) < \sum_{k=1}^{t} \sum_{y=0}^{x_k^0} c_k(y)\,.$$

A contradiction. ||

It is not true that every pure strategy equilibrium is a solution to (*). The following example is illustrative.

Example:

There are six primary factors. The costs are:

$$c_1(x_1) = x_1^2,\ c_2(x_2) = x_2^2,\ c_3(x_3) = c_4(x_4) = c_5(x_5) = 0,\ c_6(x_6) = 1\,.$$

Player 1 has two pure strategies: $\{1,3\}$ and $\{2,4\}$. Player 2's pure strategies are $\{1,5\}$ and $\{2,6\}$. The resulting normal form is:

	$\{1,5\}$	$\{2,6\}$
$\{1,3\}$	4, 4	1, 2
$\{2,4\}$	1, 1	4, 5

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References

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$= 0,\ c_6(x_6) = 1$.

2's pure strategies are

The off-diagonal elements are both pure-strategy equilibria. (1, 2) does not solve (*). The randomized strategy pair (2/3, 1/3) for player 1 and (1/2, 1/2) for player 2 is also a Nash equilibrium.

References

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