Alternative description

Note Title 3/31/2005

$\mathcal{P}_i = \{\text{paths from } s_i \to t_i\}$

$\mathcal{P} = \bigcup \mathcal{P}_i$

$f : \mathcal{P} \to \mathbb{R}^+$

$$\sum_{p \in \mathcal{P}_i} f_p = r_i$$

requirements satisfied

$$f_e = \sum_{P \ni e} f_P$$ flow on e

$l_e(f_e)$ = latency of e

It is assumed that l_e is continuous and non-decreasing.

$$l_p(f) = \sum_{e \in P} l_e(f_e)$$

Latency of P. "time taken to traverse P".

$$C(f) = \sum_{P \in \mathcal{P}} l_P(f)\, f_P$$

$$= \sum_{e \in E} l_e(f_e)\, f_e$$

Total cost of flow f

f is a <u>Nash</u> flow if

$\forall i, \forall P_1, P_2 \in \mathcal{P}_i, \delta \in [0, f_{P_1}]$

$l_{P_1}(f) \leq l_{P_2}(\tilde{f})$

$$\tilde{f}_P = \begin{cases} f_P - \delta & P = P_1 \\ f_P + \delta & P = P_2 \\ f_P & P \neq P_1, P_2 \end{cases}$$

Equivalently:

$\forall i$ and $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$,

$l_{P_1}(f) \leq l_{P_2}(f)$.

Thus in a Nash flow every path with positive flow has same latency $L_i(f)$

$$C(f) = \sum_i L_i(f)\, r_i$$

Optimal flows: $c_e(f_e) = l(f_e) \cdot f_e$

NLP1: $\min \sum_{e \in E} c_e(f_e)$

s.t.

$$\left.\begin{aligned} \sum_{p \in P_i} f_p &= r_i \\ f_e &= \sum_{p \ni e} f_p \\ f_p &\geq 0 \end{aligned}\right] f \in \mathcal{F}$$

Assume c_e is convex $\forall e$

A flow is optimal iff $\forall i$ and $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$

$$c'_{P_1}(f) \leq c'_{P_2}(f) \qquad \circledast$$

where

$$c'_P(f) = \sum_{e \in P} \frac{d}{df_e} c_e(f_e)$$

Let $\ell^*(f_e) = (\ell_e(f_e)\, f_e)'$

$$= \ell_e(f_e) + \ell'_e(f_e)\, f_e$$

Let (G, r, ℓ) be such that $x\,\ell_e(x)$ is convex $\forall e$.

f is optimal for (G, r, ℓ)

$$\Updownarrow$$

f is Nash for (G, r, ℓ^*)

Lemma

If $f, \tilde{f}$ are both Nash then

$C(f) = C(\tilde{f})$.

Proof

$$h_e(x) = \int_0^x l_e(t)\, dt \leftarrow \text{Convex}$$

Consider

NLP2: minimise $\sum_e h_e(f_e) : f \in \mathcal{F}$

Conditions for minimum

$$C'_{p_1}(f) \le C'_{p_2}(f)$$

are those for being Nash.

If $f, \tilde{f}$ are 2 Nash flows then they are both optimal for NLP2.

If $f_e \neq \tilde{f}_e$ then

$$h(\lambda f_e + (1-\lambda)\tilde{f}_e) < \lambda h(f_e) + (1-\lambda) h(\tilde{f}_e)$$

otherwise $\lambda f + (1-\lambda)\tilde{f}$ is better than either in NLP2

$$\Rightarrow l_e(f_e) = l_e(\tilde{f}_e) \quad \forall e^{*}$$

$$\Rightarrow C(f) = C(\tilde{f})$$

$$\int_0^{\lambda A + (1-\lambda) B} l(x)\,dx = \lambda \int_0^A l(x)\,dx + (1-\lambda)\int_0^B l(x)\,dx$$

$\Rightarrow$ $l(x)$ constant on $[A, B]$

[Differentiate w.r.t. λ]

Theorem

If f is Nash for (G, r, l)

and f^* is feasible for $(G, 2r, l)$

then $C(f) \le C(f^*)$.

Proof

$$\bar{l}_e(x) = \begin{cases} l_e(f_e) & x \le f_e \\ l_e(x) & x \ge f_e \end{cases}$$

$$\sum_e \bar{l}_e(f_e^*) f_e^* - C(f^*) =$$

$$\sum_e f_e^* \left(\bar{l}_e(f_e^*) - l_e(f_e^*) \right)$$

$$\leq \sum_e l_e(f_e) f_e$$

$$= C(f)$$

We write this as

$$C(f^*) \geq \sum_p \bar{l}_p(f^*) f_p^* - C(f)$$

$$\bar{l}_p(\text{zero flow}) \geq L_i(f) \Rightarrow$$

$$\bar{l}_p(f^*) \geq L_i(f)$$

$$\sum_p \bar{l}_p(f^*) f_p^* \geq$$

$$\sum_i \sum_{p \in \mathcal{P}_i} L_i(f) f_p^* =$$

$$\sum_i L_i(f) \times 2 r_i$$

$$= 2C(f).$$

But then

$$C(f^*) \geq \sum_{P} \bar{\ell}_P(f^*) f_P^* - C(f)$$

$$\geq 2C(f) - C(f)$$

$\square$

Linear Latency Functions:

Now assume that

$$l_e(x) = a_e x + b_e$$

(a) f is Nash iff $\forall i, P, P' \in \mathcal{P}_i, f_P > 0$

$$\sum_{e \in P} a_e f_e + b_e \le \sum_{e \in P'} a_e f_e + b_e$$

(b) f is optimal iff $\forall i, P, P' \in \mathcal{P}_i, f_P > 0$

$$\sum_{e \in P} 2 a_e f_e + b_e \le \sum_{e \in P'} 2 a_e f_e + b_e$$

Lemma

f is Nash for (G, r, l) $\Longrightarrow$

(i) $f/2$ is optimal for $(G, r/2, l)$

(Compare (a), (b) on previous page)

(ii) Marginal cost of increasing flow on P w.r.t. $f/2$ equals latency of P w.r.t. f

(marginal cost l^* for $l = ax + b$ is

$$\big((ax+b)\,x\big)' = 2ax + b \;)$$

i.e. $l^*(x/2) = l(x)$

Lemma

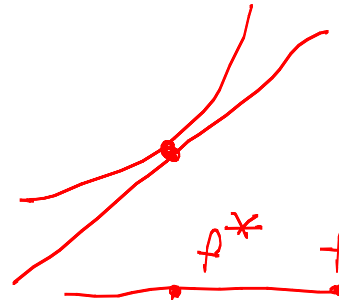
f^* is optimal for (G, r, ℓ)

f is feasible for $(G, (1+\delta)r, \ell)$

$$\Rightarrow \quad C(f) \geq C(f^*) + \delta \sum_i L_i^*(f^*) r_i$$

$L_i^*(f^*)$ is the minimum marginal cost of increasing flow along an $s_i - t_i$ path w.r.t. f^*.

Proof



$$C(f) = \sum_e l_e(f_e) f_e$$

$$\geq \sum_e l_e(f_e^*) f_e^* + \sum_e (f_e - f_e^*) l_e^*(f_e^*)$$

$$= C(f^*) + \sum_i \sum_{p \in \mathcal{P}_i} l_p^*(f^*)(f_p - f_p^*)$$

[$L_i^*(f^*) \leq l_p^*(f^*)$, $\forall i, p \in \mathcal{P}_i$ and equality holds, unless $f_p^* = 0$]

$$\geq C(f^*) + \sum_i L_i(f^*) \sum_{p \in \mathcal{P}_i} (f_p - f_p^*)$$

$$= C(f^*) + \delta \sum_{i=1}^{k} L_i^*(f^*) r_i \qquad \square$$

Theorem

If f is Nash for (G, r, l) then

$$C(f) \le \frac{4}{3} C(\text{opt})$$

Proof

Take $\delta = 1$ in previous lemma.

Let f^* be any feasible flow.

$$C(f^*) \ge C(f/2) + \sum_i L_i^*(f/2) \frac{r_i}{2}$$

$$= C(f/2) + \frac{1}{2} \sum_i L_i(f) \, r_i$$

$$= C(f/2) + \frac{1}{2} C(f)$$

$$C(f/2) = \sum_e \frac{1}{4} a_e f_e^2 + \frac{1}{2} b_e f_e$$

$$\geq \frac{1}{4}\left(\sum_e a_e f_e^2 + b_e f_e\right)$$

$$= \frac{1}{4} C(f).$$

$\square$