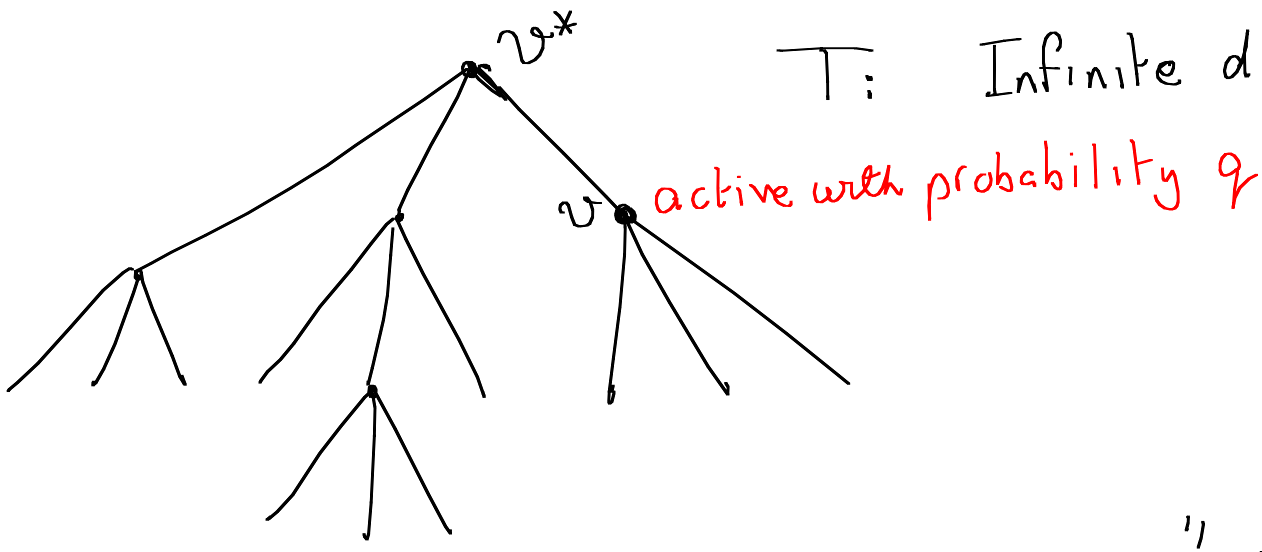
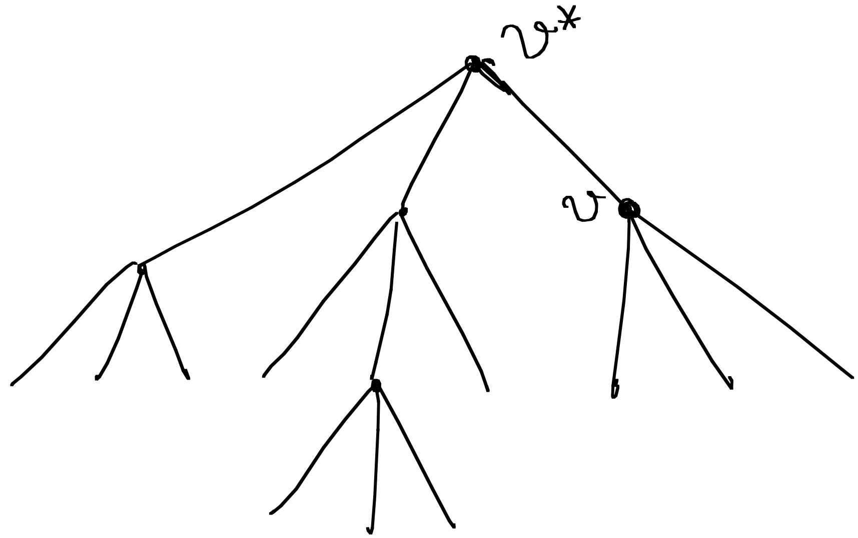
# Query Incentive Networks



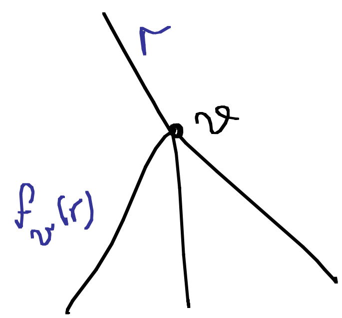

$T$: Infinite $d$-ary tree

$v^*$ seeks some "answer" to a "query".

Node $v$ holds this answer with probability $1-p$.



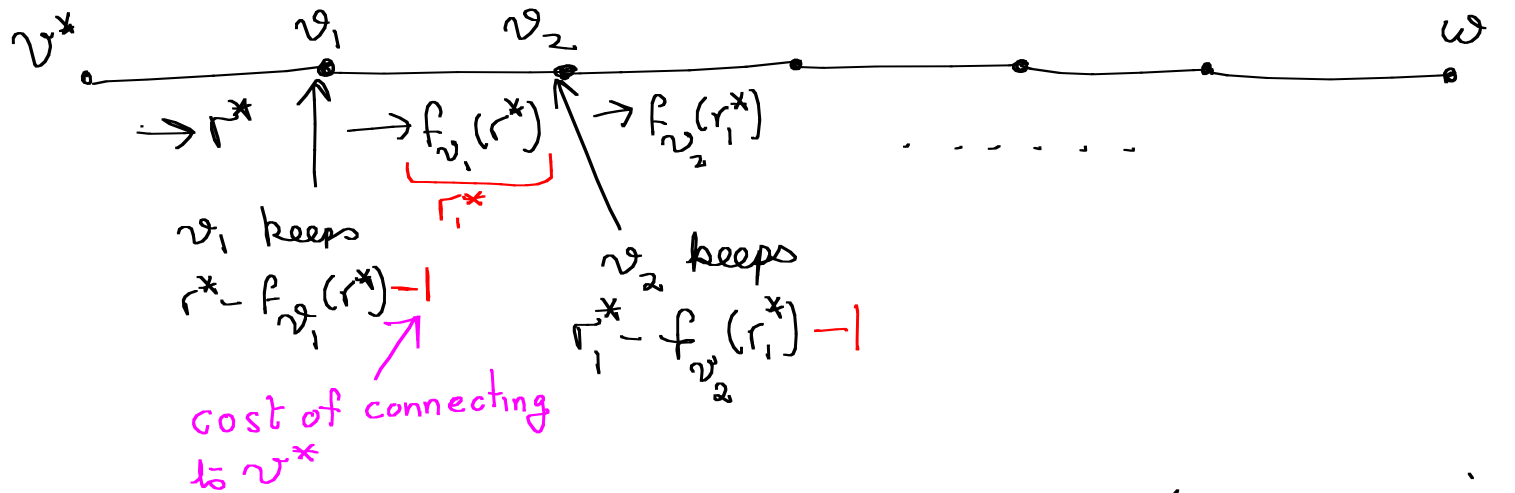

$v^*$ attaches "utility" $r^* \in \underbrace{\{1, 2, 3, \ldots\}}_{N}$ to obtaining this answer.




If $v$ is offered a reward $r \in N$ and it does not hold the answer then it offers reward $f_v(r) < r$ to its children.

Query is passed down, so long as $f_v(r) > 0$.

Suppose some node is found that contains answer. $v^*$ selects one such $w$.



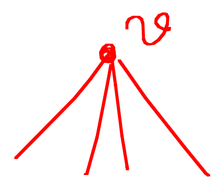

Each vertex $v$ will choose $f_v$ to try to maximise its expected payoff.

We study Nash Equilibria of this game.

We estimate value of $r^*$ needed to get the answer with probability $\sigma > 0$.

$T' =$ sub-tree of active nodes

$=$ tree produced by Galton-Watson branching process




$i$ desendants with probability

$\binom{d}{i} q^i (1-q)^{d-i}$.

$n = \frac{1}{1-p} =$ "rarity" of answer.

Roughly 1 in $n$ nodes contain answer

$b = qd =$ expected branching factor.

$b < 1 \Rightarrow |T'| < \infty$ with probability one

$b > 1 \Rightarrow Pr(|T'| < \infty) = e_{q,d} < 1$.

Theorem

(i) $b < 2 \Rightarrow$ need $r^* = \Omega(n^c)$ to obtain answer
constant probability $\sigma$ of getting answer

$c$ depends on $2-b$.

(ii) $b > 2 \Rightarrow$ need $r^* = O(\log n)$ to obtain answer
with constant probability $\sigma < 1 - e_{q,d}$.

$\mathbf{f} = \{ f_v : v \in T \}$ = set of all reward functions

= "strategies" of players

$\alpha_v(\mathbf{f}, x)$ = probability the subtree of $T'$ below $v$ yields answer if $v$ offers reward $x$.

$= 1 - \beta_v(\mathbf{f}, x)$.

$$\beta_v(\mathbf{f}, x) = \prod_{w \text{ child of } v} \Big(1 - \underset{\uparrow}{q}\big(1 - \underbrace{p\,\beta_w(\mathbf{f}, f_w(x))}_{}\big)\Big)$$

Pr(w active)

Pr(w does not deliver info)

= Pr(w does not have it) ×

Pr(sub-tree does not produce)

Now define Nash Equilibrium $\mathbf{g} = \{ g_v : v \in T \}$

(i) $g_v(1) = 0 \quad \forall v$

(ii) Assume $g_v(x)$ has been defined $\forall x < r,\ v \in T$

$g_v(r) =$ value of $x < r$ that maximises

$$(r - x - 1)\,\alpha_v(\mathbf{g}, x).$$

Expected reward if on path $v^* \rightarrow$ answer.

By construction $g_v = g$ (independent of $v$).

(iii) $g(2) = 1.$

Thm

$\mathbf{g}$ is a Nash Equilibrium.

Proof

Given $r^*$ and $\mathbf{f}$ define events:

A: $v$ gets part of reward.

B: info is in sub-tree below $v$ and is passed to $v$.

C: query reaches $v$.

D: $v$ holds answer.

$$Y_{\mathbf{f},r} = \sum_{\underset{\uparrow}{A',B',C',D'}} E\left(Y_{\mathbf{f},r} \,\middle|\, A',B',C',D'\right) P_r(A',B',C',D')$$

A or $\bar{A}$, etc.

$$E(Y_{\mathbf{f},r} \mid \bar{A}) = E(Y_{\mathbf{f},r^*} \mid \bar{C}) = 0$$

$$\Pr(A, \bar{B}) = 0.$$

So.

$$E(Y_{\mathbf{f},r}) = \overbrace{E(Y_{\mathbf{f},r} \mid A, B, C, D)}^{r-1} \times \Pr(A, B, C, D) \; +$$

$$\underbrace{E(Y_{\mathbf{f},r} \mid A, B, C, \bar{D})}_{r - f_v(r) - 1} \times \underbrace{\Pr(A, B, C, \bar{D})}_{\Pr(A \mid B, C, \bar{D}) \times \underbrace{\Pr(B \mid C, \bar{D})}_{\alpha_v(\mathbf{f}, f_v(r))} \times \Pr(C, \bar{D}).}$$

$r = \rho_v(\mathbf{f}, r) =$ reward to $v$.

Choice of $\mathbf{g}$ maximises $E(Y_{\mathbf{f},r} \mid A, B, C, \bar{D}) \times \Pr(B \mid C, \bar{D})$, over choice of $f_v(r)$.

Remaining terms in $E(Y_{\mathbf{f},r})$ are not affected by choice of $f_v(r)$.

Any deviation from $f_v(r)$ cannot result in improvement for $v$. $\square$

## Uniqueness of Nash

Assume:

(a) $f_v(2) = 1, \quad \forall v \in T$

(b) $(p, q)$ is not the root of a bivariate polynomial with rational coefficients. [Generic]

Reward $r$ is reachable at $v$ if $\exists r^*$ such that $v$ gets $r$ with positive probability.

**Thm**

$\mathbf{f}$ is N.E. & (a), (b) hold. Then

$$f_v(r) = g_v(r)$$

for all $v \in T$ & reachable $r$ at $v$.

Proof

$f_\nu(1) = 0 = g_\nu(1)$.

Let $r > 2$ be smallest reachable reward for which $f_\nu(r) \neq g_\nu(r)$.

Payoff to $\nu$ from offering $x$ is $\psi + \xi(r-x-1)\,\alpha_\nu(\mathbf{f}, x)$

[ $\psi, \xi$ do not depend on $x$ ].

$= \psi + \xi(r-x-1)\,\alpha_\nu(\mathbf{g}, x)$

Setting $x = g_\nu(r)$ maximises

& setting $x = f_\nu(r)$ maximises

But then

$$(r - f_\nu(r) - 1)\,\alpha_\nu(\mathbf{f}, f_\nu(r)) = (r - g_\nu(r) - 1)\,\alpha_\nu(\mathbf{g}, g_\nu(r))$$

Contradicts (b).

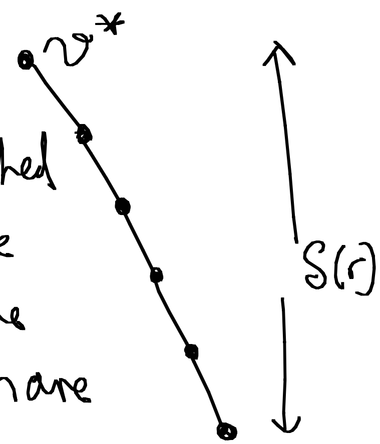
$\square$

$R_\sigma(n,b)$ = smallest reward needed to yield an answer with probability $\sigma$.

$\delta(r)$ = # times we iterate $g$ to get $0$.

$$\underbrace{g(g(\cdots\ g}_{\delta(r)\text{ times}}(r))\cdots) = 0.$$

Level reached if all these nodes continue and none have answer




So we run branching process until either (i) dies out, (ii) finds node with answer or (iii) reaches level $\delta(r)$.

$\hat{\phi}_j$ = probability that no active node in first $j$ levels has answer, given that $v^*$ does not.

(i) $\hat{\phi}_{\delta(r)} = \beta_{v^*}(\mathbf{g}, r)$

(ii) $\hat{\phi}_{j+1} = \left(1 - q\left(1 - p\hat{\phi}_j\right)\right)^d$

$u_j = \min\{r : \delta(r) > j-1\}$

Claim: If $u_j$ exists then $\delta(u_j) = j$.

$$\delta(g(u_j)) \leq j-1 \quad \& \quad \delta(u_j) = 1 + \delta(g(u_j)).$$

Thus $u_1 < u_2 < \cdots < u_j < u_{j+1} < \cdots$

## Existence of $u_j$

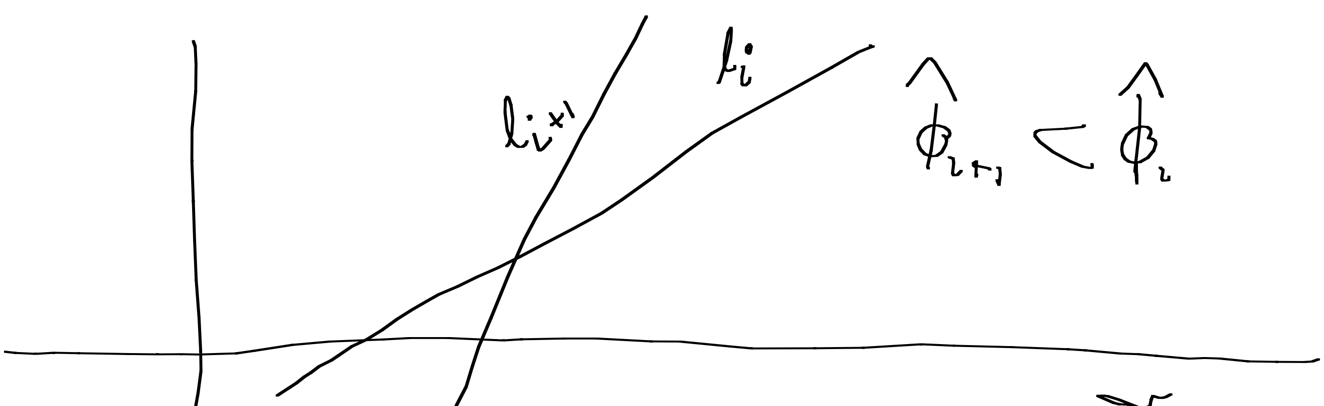
$u_1 = 1$ & $u_2 = 2$.

Assume that we have defined $u_1, u_2, \ldots, u_j$.

Only possible optimal rewards are of form $u_i$.
We can always reduce offered reward to a $u_i$ without decreasing probability of success.

Let $\quad l_i(r) = (r - u_i - 1)(1 - \hat{\phi}_i)$

Expected payoff to root if it offers $u_i$.




$\hat{\phi}_{i+1} < \hat{\phi}_i$

Assume inductively that

$$r \geq u_j \Rightarrow l_{j-1}(r) > l_{j-2}(r) > \cdots > l_1(r).$$

Now define

$$u_{j+1} = \lceil y_{j+1} \rceil$$

where

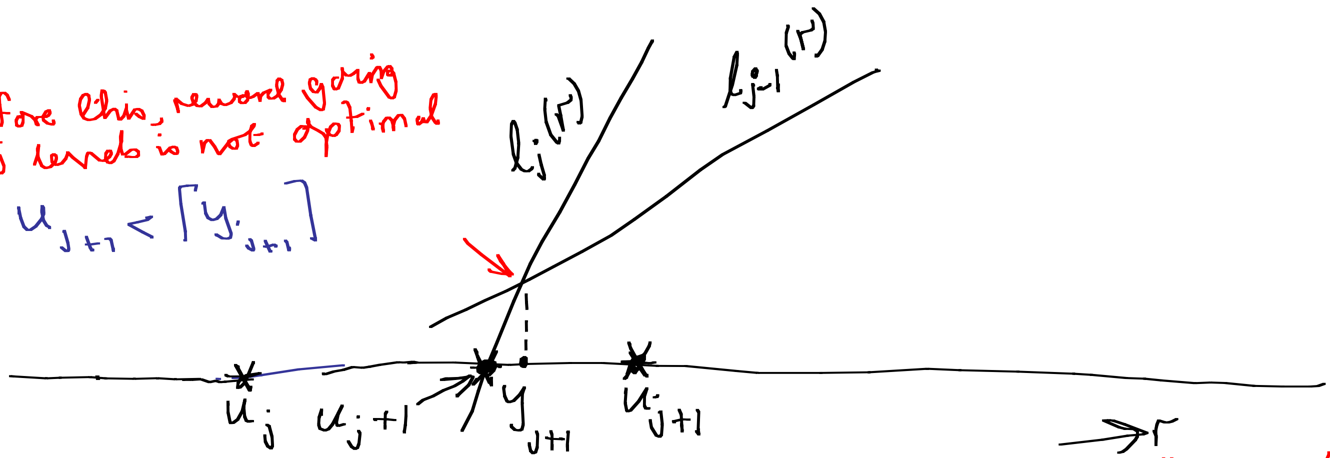
$$l_j(y_{j+1}) = l_{j-1}(y_{j+1})$$

Note: $l_j(1+u_j) = 0 < l_{j-1}(1+u_j)$ and so $y_{j+1} > u_j$.

COMPLETES INDUCTION

Before this, reward going to $j$ levels is not optimal

$\Rightarrow u_{j+1} < \lceil y_{j+1} \rceil$




From this point on reward that "reaches" $j$ levels is better than reward that reaches $j-1$ levels

$\Rightarrow u_{j+1} \leq \lceil y_{j+1} \rceil$

More parameters:

$$\Delta_j' = y_j - y_{j-1} \quad \& \quad \Delta_j = u_j - u_{j-1}$$

$$1 + \frac{\Delta_j}{\Delta_{j+1}' - 1} = \frac{1 - \hat{\phi}_j}{1 - \hat{\phi}_{j-1}}$$

$$t(x) = (1 - q(1 - px))^d$$

$$\hat{\phi}_j = t(\hat{\phi}_{j-1}).$$

Remember $pb > 1.$

# CLAIMS

(1) If $\frac{1}{dn} < \epsilon < 1$ and $x \in [1-\epsilon, 1]$ then

$$t'(x) \in [pb(1-2bd\epsilon), pb]$$

(2) $1 - \frac{b}{n} \leq t(1) \leq 1 - \frac{1}{dn}$

(3) Suppose $pb(1-2bd\epsilon) > 1$ and $0 < \gamma_0 < \gamma_1 \leq \epsilon$.

min. needed

$$t^{N(\gamma_0,\gamma_1)}(1-\gamma_0) \leq 1-\gamma_1$$

$$N(\gamma_0,\gamma_1) = \Theta(\log(\gamma_1/\gamma_0)).$$

## $b<2$

Choose $\sigma_0 \ll \sigma$ so that $pb(1-2bd\sigma_0) > 1$

Choose $\kappa_0 > \frac{b}{2-b}$

$$I_1 = \{ j : \hat{\phi}_j \geq 1 - \frac{\kappa_0}{n} \}$$

$$I_2 = \{ j : 1 - \sigma_0 \leq \hat{\phi}_j < 1 - \frac{\kappa_0}{n} \}$$

## Lemma

Claim 3 $\Longrightarrow$ $|I_1| = O(1)$ $\quad$ $\left[ 1 - \frac{b}{n} \leq \hat{\phi}_1 \leq 1 - \frac{1}{dn} \right.$

$|I_2| = \Theta(\log n)$

Lemma

$\exists\, b_1 < 2$ such that $\dfrac{1-\hat{\phi}_{j+1}}{1-\hat{\phi}_j} \leq b_1$ for $j \in I_2$.

Proof

$$\frac{1-\hat{\phi}_{j+1}}{1-\hat{\phi}_j} = \frac{1-t(x)}{1-x} \qquad x = \hat{\phi}_j \in \left[1-\sigma_0,\ 1-\frac{\kappa_0}{n}\right]$$

$$= \frac{1-t(1)}{1-x} + \frac{t(1)-t(x)}{1-x}$$

$$= \frac{1-t(1)}{1-x} + t'(y) \qquad \text{where } y \in [x, 1]$$

Claim 1 $\Rightarrow$ $t'(y) \leq pb$

$$\frac{1-t(1)}{1-x} \leq \frac{1-\left(1-\frac{b}{n}\right)}{1-\left(1-\frac{\kappa_0}{n}\right)} = \frac{b}{\kappa_0}.$$

Put $b_1 = pb + \frac{b}{k_0} \geq \frac{1-\hat{\phi}_{j+1}}{1-\hat{\phi}_j}$.

$$\downarrow$$

$< pb + 2 - b$

$\leq 2$.

## Theorem

$\exists\, c > 0$ such that if $\hat{\phi}_j < 1-\sigma$ then $u_j \geq n^c$

$\Rightarrow R_\sigma(n, b) \geq n^\sigma$.

## Proof

$$\Delta_j = u_j - u_{j-1}$$

$$= \frac{\Delta_j}{\Delta_{j-1}} \cdot \frac{\Delta_{j-1}}{\Delta_{j-2}} \cdots \cdot \frac{\Delta_3}{\Delta_2} \cdot u_1$$

$u_1 = 1$

$u_2 = 1$

$$c_0 = \frac{1}{b_1 - 1}.$$

$$j \in I_2 \Rightarrow \frac{\Delta_{j+1}}{\Delta_j} \geq \frac{\Delta'_{j+1} - 1}{\Delta_j} = \frac{1}{\frac{1-\hat{\phi}_{j+1}}{1-\hat{\phi}_j} - 1} \geq \frac{1}{b_1 - 1} > 1.$$

$$u_j \geq \Delta_j = u_1 \prod_{i=3}^{j} \frac{\Delta_i}{\Delta_{i-1}} \geq \prod_{i \in I_2} \frac{\Delta_i}{\Delta_{i-1}}$$

$$\geq c_0^{|I_2|} \geq c_0^{\tau \log n} = n^{\tau \log c_0}.$$

$\square$

## $b > 2$

Choose $\sigma_0$ so that $pb(1-2bd\sigma_0) > 2$.

$$J_1 = \{ j \geq 1 : \hat{\phi}_j \geq 1 - \sigma_0 \}$$

$$J_2 = \{ j : 1 - \sigma \leq \hat{\phi}_j < 1 - \sigma_0 \}$$

Claim 3 $\Longrightarrow$ $|J_1| = \Theta(\log n)$.

**Lemma**

$\exists\, b_2 > 2$ such that $j \in J_1 \Rightarrow \dfrac{1-\hat{\phi}_{j+1}}{1-\hat{\phi}_j} \geq b_2$

**Proof**

$$x = \hat{\phi}_j \in \left[1-\sigma_0,\ 1 - \frac{1}{dn}\right].$$

$$\frac{1-\hat{\phi}_{j+1}}{1-\hat{\phi}_j} = \frac{1-f(x)}{1-x} \geq \frac{f(1)-f(x)}{1-x} = f'(y) \geq \underbrace{pb(1-2bd\sigma_0)}_{b_2} > 2.$$

$\square$

**Lemma**

$|J_2| = O(1)$.

**Proof**

$$\bar{t}(x) = (1 - q(1-x))^d = (qx + (1-q))^d$$

= generating function for the distribution of number of children in our branching process.

$$\bar{t}(e_{q,d}) = e_{q,d} \in [0,1).$$

$$t(x) < \bar{t}(x) < x \qquad x \in (e_{q,d}, 1).$$

$\eta(\epsilon, \epsilon') = \#$ iterations of $\bar{t}$ needed to reduce $1-\epsilon'$ to $e_{q,d}+\epsilon$.

$$|J_2| \leq \eta(1 - e_{q,d} - \sigma, \sigma_0).$$

$\square$

## Theorem

$\exists$ constant $c'$ so that $\hat{\phi}_j < 1 - \sigma$ and $u_j \leq c' \log n$.

Thus $R_\sigma(n, b) = O(\log n)$.

## Proof

Let $j_i := \max J_i$, $i = 1, 2$.

Claim: $\Delta_j \leq \dfrac{2(b_2 - 1)}{b_2 - 2} = O(1)$ for $j \in J_1$.

Induction: $\Delta_2 = 2$.

$$\frac{\Delta'_{j+1} - 1}{\Delta_j} = \frac{1}{\dfrac{1 - \hat{\phi}_{j+1}}{1 - \hat{\phi}_j} - 1} \leq \frac{1}{b_2 - 1}$$

So

$$\Delta_{j+1} \leq \Delta'_{j+1} + 1 \leq \frac{\Delta_j}{b_2 - 1} + 2 \leq \frac{2(b_2 - 1)}{b_2 - 2}.$$

Hence $\hat{\phi}_{j+1} < 1 - \sigma_0$ and $u_{J_1} = O(\log n)$.

$x - t(x) > 0$ on $[1-\sigma, 1-\sigma_0]$ and so achieves a positive constant $\mu$.

$J_2$ consists of $O(1)$ iterations in which

$$\frac{\Delta_{j+1}}{\Delta_j} \leq \frac{\Delta'_{j+1} - 1}{\Delta_j} + 1$$

$$= 1 + \frac{1}{\frac{1-\hat{\phi}_{j+1}}{1-\hat{\phi}_j} - 1} = 1 + \frac{1}{\hat{\phi}_j - \hat{\phi}_{j+1}}$$

$$= 1 + \frac{1}{\hat{\phi}_j - t(\hat{\phi}_j)} \leq 1 + \frac{1}{\mu}.$$

So $u_{J_2} = O(\log n)$. $\square$