Appendix 2: Existence of Equilibria in Finite Games

We give a proof of Nash's Theorem based on the celebrated Fixed Point Theorem of L. E. J. Brouwer. Given a set C and a mapping T of C into itself, a point $\boldsymbol{z} \in C$ is said to be a fixed point of T, if $T(\boldsymbol{z}) = \boldsymbol{z}$.

Brouwer's Fixed Point Theorem. *Let C be a nonempty, compact, convex set in a finite dimensional Euclidean space, and let T be a continuous map of C into itself. Then there exists a point $\boldsymbol{z} \in C$ such that $T(\boldsymbol{z}) = \boldsymbol{z}$.*

The proof is not easy. You might look at the paper of K. Kuga (1974), "Brower's fixed point Theorem: An Alternate Proof", *SIAM Journal of Mathematical Analysis,* **5**, 893-897. Or you might also try Parthasarathy and Raghavan (1971), Chapter 1.

Now consider a finite n-person game with the notation of Section III.2.1. The pure strategy sets are denoted by $X_1, \ldots, X_n$, with X_k consisting of $m_k \geq 1$ elements, say $X_k = \{1, \ldots, m_k\}$. The space of mixed strategies of Player k is given by X_k^*,

$$X_k^* = \{\boldsymbol{p}_k = (p_{k,1}, \ldots, p_{k,m_k}) : p_{k,i} \geq 0 \text{ for } i = 1, \ldots, m_k, \text{ and } \textstyle\sum_{i=1}^{m_k} p_{k,i} = 1\}. \qquad (1)$$

For a given joint pure strategy selection, $\boldsymbol{x} = (i_1, \ldots, i_n)$ with $i_j \in X_j$ for all j, the payoff, or utility, to Player k is denoted by $u_k((i_1, \ldots, i_n)$ for $k = 1, \ldots, n$. For a given joint mixed strategy selection, $(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$ with $\boldsymbol{p}_j \in X_j^*$ for $j = 1, \ldots, n$, the corresponding expected payoff to Player k is given by $g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$,

$$g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) = \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} p_{1,i_1} \cdots p_{n,i_n} u_k(i_1, \ldots, i_n). \qquad (2)$$

Let us use the notation $g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n | i)$ to denote the expected payoff to Player k if Player k changes strategy from p_k to the pure strategy $i \in X_k$,

$$g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n | i) = g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{k-1}, \boldsymbol{\delta}_i, \boldsymbol{p}_{k+1}, \ldots, \boldsymbol{p}_n). \qquad (3)$$

where $\boldsymbol{\delta}_i$ represents the probability distribution giving probability 1 to the point i. Note that $g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$ can be reconstructed from the $g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n | i)$ by

$$g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) = \sum_{i=1}^{m_k} p_{k,i} g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n | i) \qquad (4)$$

A vector of mixed strategies, $(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$, is a strategic equilibrium if for all $k = 1, \ldots, n$, and all $i \in X_k$,

$$g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n | i) \leq g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n). \qquad (5)$$

Theorem. *Every finite n-person game in strategic form has at least one strategic equilibrium.*

Proof. For each k, X_k^* is a compact convex subset of m_k dimensional Euclidean space, and so the product, $C = X_1^* \times \cdots \times X_n^*$, is a compact convex subset of a Euclidean space of dimension $\sum_{i=1}^n m_i$. For $\boldsymbol{z} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) \in C$, define the mapping $T(\boldsymbol{z})$ of C into C by

$$T(\boldsymbol{z}) = \boldsymbol{z}' = (\boldsymbol{p}_1', \ldots, \boldsymbol{p}_n') \tag{6}$$

where

$$p_{k,i}' = \frac{p_{k,i} + \max(0, g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n|i) - g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n))}{1 + \sum_{j=1}^{m_k} \max(0, g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n|j) - g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n))}. \tag{7}$$

Note that $p_{k,i} \geq 0$, and the denominator is chosen so that $\sum_{i=1}^{m_k} p_{k,i}' = 1$. Thus $\boldsymbol{z}' \in C$. Moreover the function $f(\boldsymbol{z})$ is continuous since each $g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$ is continuous. Therefore, by the Brouwer Fixed Point Theorem, there is a point, $\boldsymbol{z}' = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \in C$ such that $T(\boldsymbol{z}') = \boldsymbol{z}'$. Thus from (7)

$$q_{k,i} = \frac{q_{k,i} + \max(0, g_k(\boldsymbol{z}'|i) - g_k(\boldsymbol{z}'))}{1 + \sum_{j=1}^{m_k} \max(0, g_k(\boldsymbol{z}'|j) - g_k(\boldsymbol{z}'))}. \tag{8}$$

for all $k = 1, \ldots, n$ and $i = 1, \ldots, m_n$. Since from (4) $g_k(\boldsymbol{z}')$ is an average of the numbers $g_k(\boldsymbol{z}'|i)$, we must have $g_k(\boldsymbol{z}'|i) \leq g_k(\boldsymbol{z}')$ for at least one i for which $q_{k,i} > 0$, so that $\max(0, g_k(\boldsymbol{z}'|i) - g_k(\boldsymbol{z}')) = 0$ for that i. But then (8) implies that $\sum_{j=1}^{m_k} \max(0, g_k(\boldsymbol{z}'|j) - g_k(\boldsymbol{z}')) = 0$, so that $g_k(\boldsymbol{z}'|i) \leq g_k(\boldsymbol{z}')$ for all k and i. From (5) this shows that $\boldsymbol{z}' = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$ is a strategic equilibrium. ■

Remark. From the definition of $T(\boldsymbol{z})$, we see that $\boldsymbol{z} = (p_1, \ldots, p_n)$ is a strategic equilibrium if and only if $\boldsymbol{z}$ is a fixed point of T. In other words, the set of strategic equilibria is given by $\{\boldsymbol{z} : T(\boldsymbol{z}) = \boldsymbol{z}\}$. If we could solve the equation $T(\boldsymbol{z}) = \boldsymbol{z}$ we could find the equilibria. Unfortunately, the equation is not easily solved. The method of iteration does not ordinarily work because T is not a contraction map.