Nash equilibria of a network creation game

n agents = vertices build a graph between them.

Vertex v chooses a set $S_v \subseteq V \setminus \{v\}$ and creates edges (v,w), $w \in S_v$.

$\vec{S} = (S_1, S_2, \ldots, S_n)$ and $G = G(\vec{S})$ is graph created.

$$\mathrm{Cost}(v, \vec{S}) = \alpha |S_v| + \sum_{w \in V \setminus \{v\}} \delta(v,w)$$

shortest distance $v \to w$.

$\vec{S}$ is a Nash Equilibrium if for all i, $\tilde{S}_i$

$$\mathrm{Cost}(i, S_1, S_2, \ldots, S_i, \ldots, S_n) \le \mathrm{Cost}(i, S_1, S_2, \ldots, \tilde{S}_i, \ldots, S_n)$$

$G(\vec{S})$ is "Equilibrium Graph"

Multi-person game introduced by

Fabrikant, Luthra, Maneva, Papadimitriou, Shenker PODC, 03

Here we prove some theorems from

Albers, Eilts, Even-Dar, Mansour, Roditty.

$$\text{Cost}(\vec{S}) = \sum_{v} \text{Cost}(v, \vec{S})$$

Let S^* minimise Cost.

Aim is bound

$$\max_{\vec{S}} \frac{\text{Cost}(\vec{S})}{\text{Cost}(S^*)}$$

Price of Anarchy

Π_α

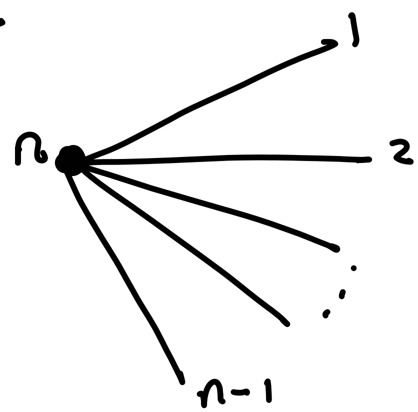
$\vec{S}$ is N.E.

Simple Observations

(i) $\alpha \leq 1$

$G(S^*) = K_n$ and $G(\vec{S}) = K_n$ for all N.E.

(ii) $\alpha \geq 1$

$S^* =$ 

S^* is also N.E.

(iii) $\alpha \geq n^2$

Every spanning tree is N.E.:

$$\Pi_\alpha \leq \frac{\alpha(n-1) + n^3}{\alpha(n-1) + 2n} = 1 + o(1).$$

Thm 1

If $\alpha \geq 12 n \log n$ then

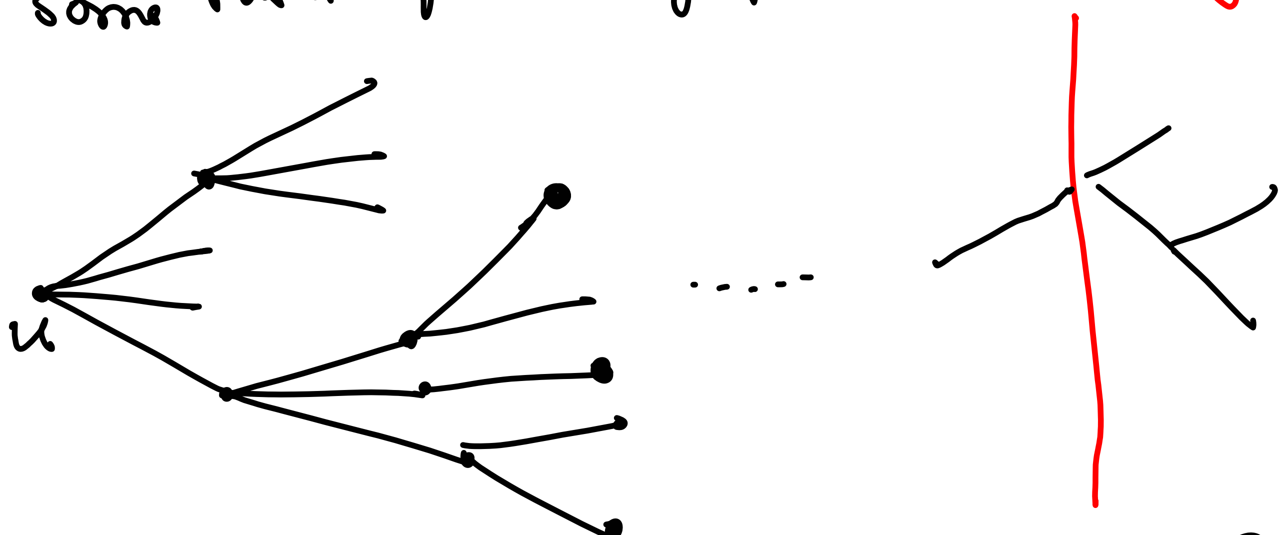
(i) $\Pi_\alpha \leq 1 + \frac{6 n \log n}{\alpha} \leq 3/2$.

(ii) Every equilibrium graph is a tree.

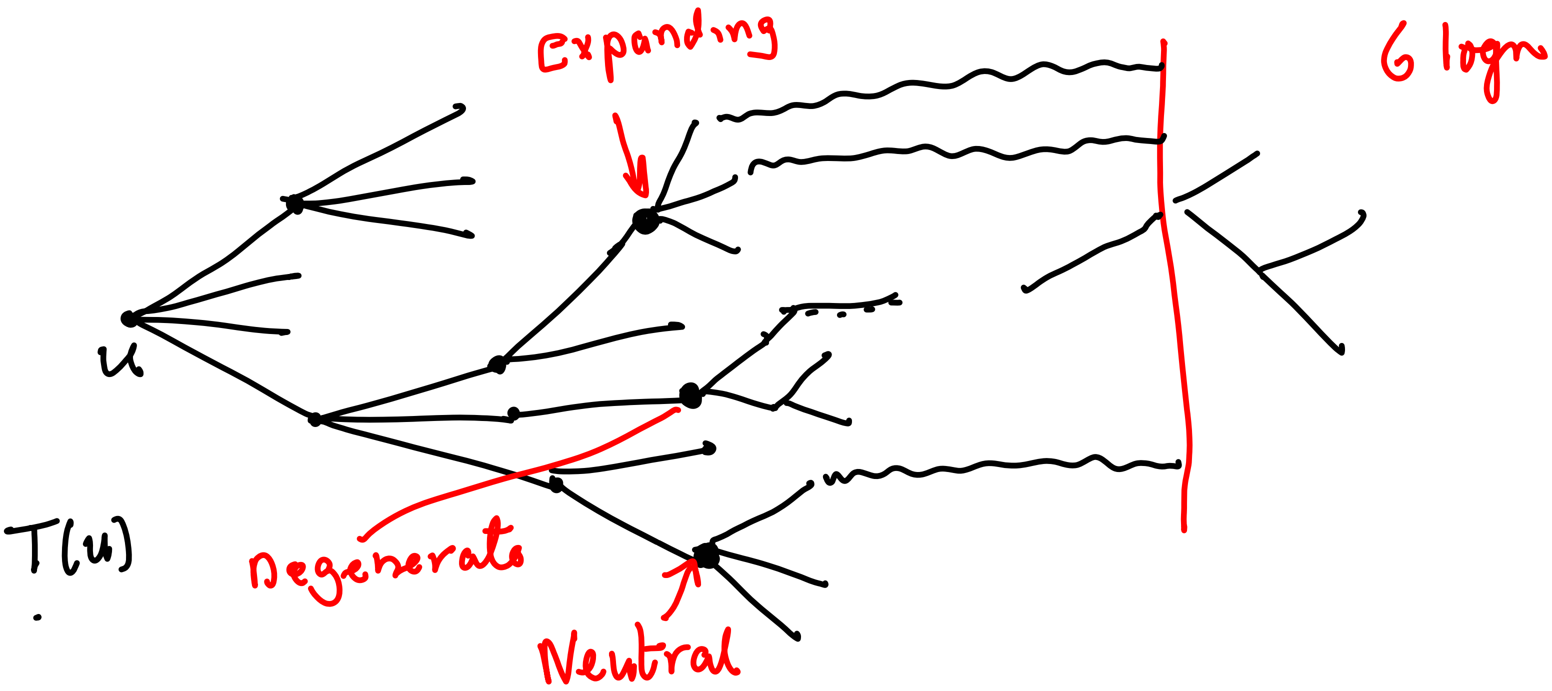
Proof

G is some fixed equilibrium graph.

Boundary = distance $6 \log n$

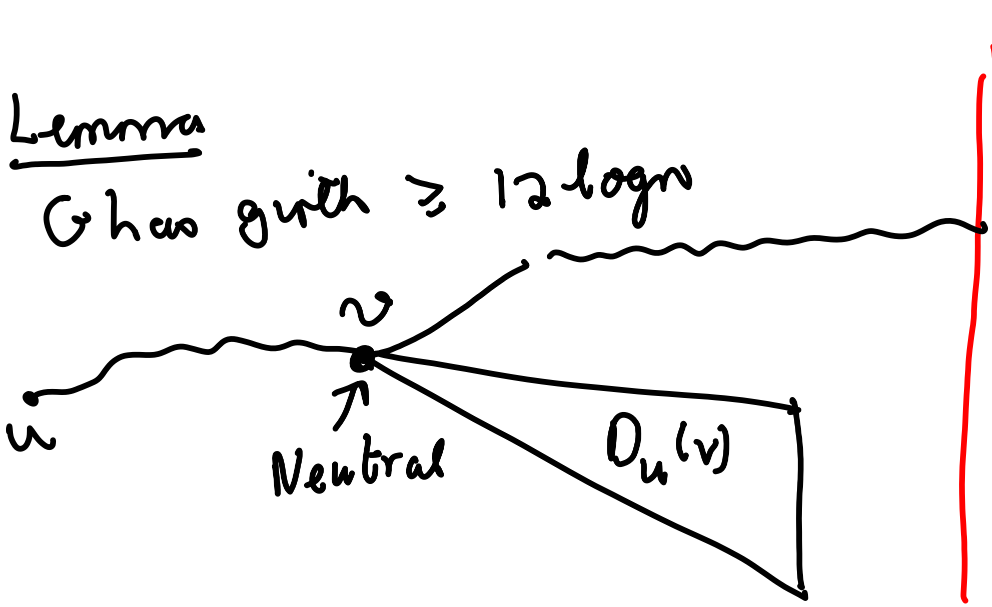


$T(u)$ is a shortest path (BFS) tree in G.

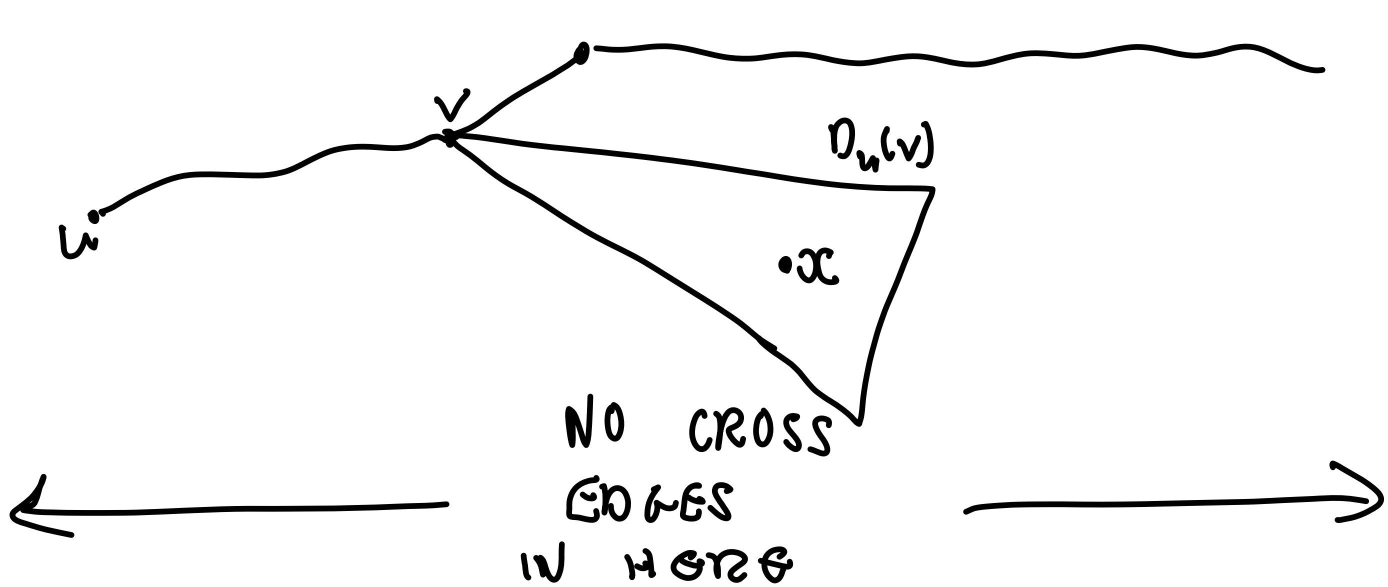


Lemma

G has girth $\geq 12 \log n$

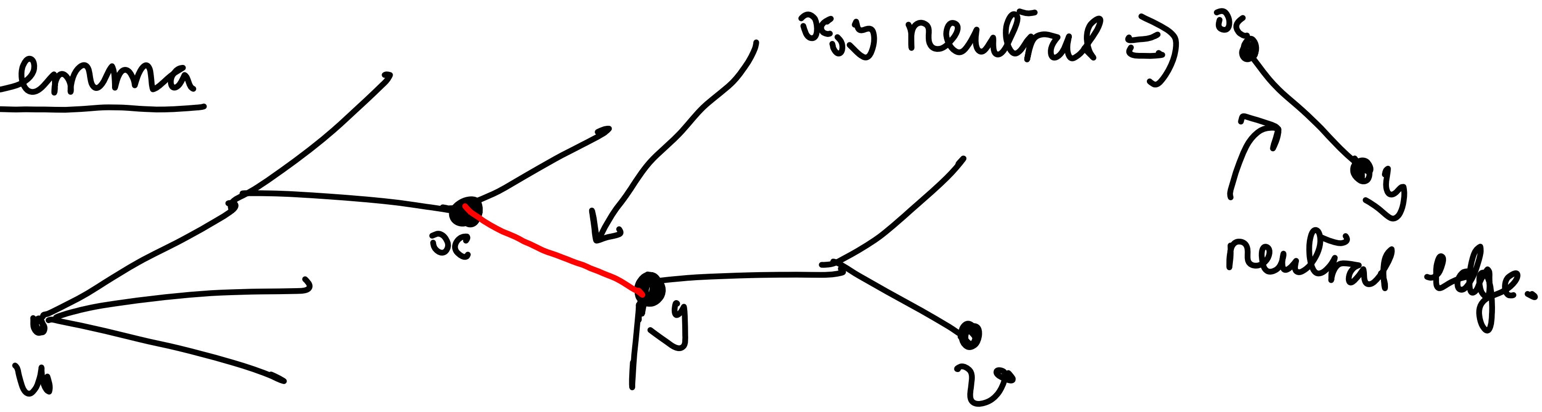


All paths in G from $D_u(v)$ to $V \setminus D_u(v)$ go through v.



& NO PATH TO BOUNDARY

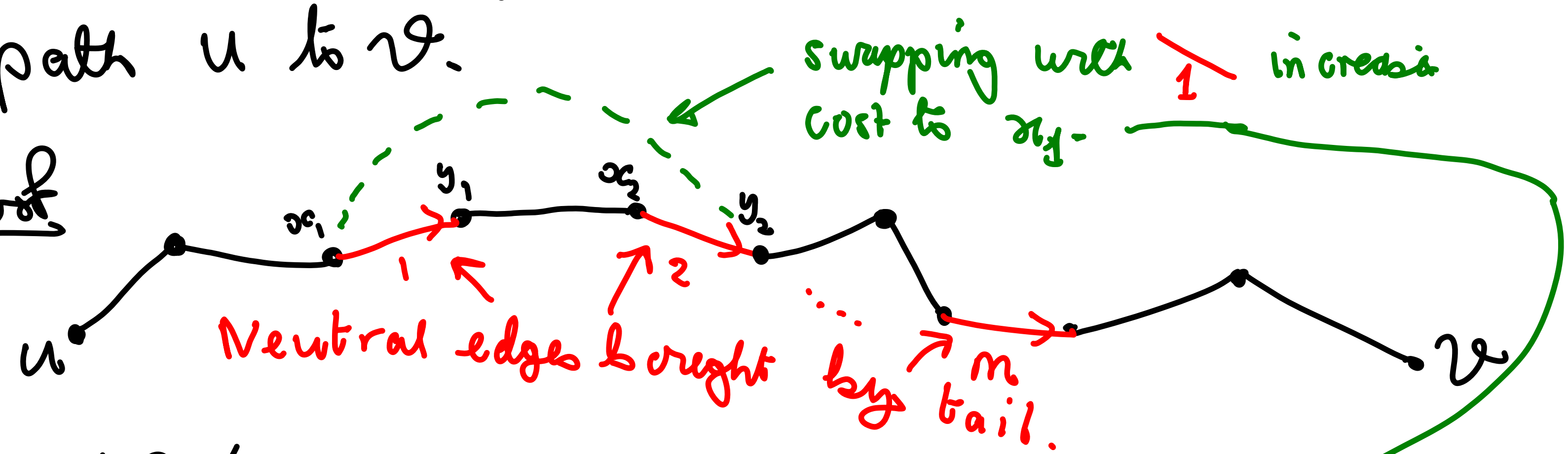
Lemma



$T(u)$

Girth $\geq 12 \log n \Rightarrow \leq 2 \log n$ neutral edges on path u to v.

Proof



$$n_j = |D_u(y_j)| \geq n_{j+1} + n_{j+2} + \cdots + n_m$$

$$\Rightarrow n_j \geq 2^{m-j+1} \Rightarrow m \leq \log n.$$

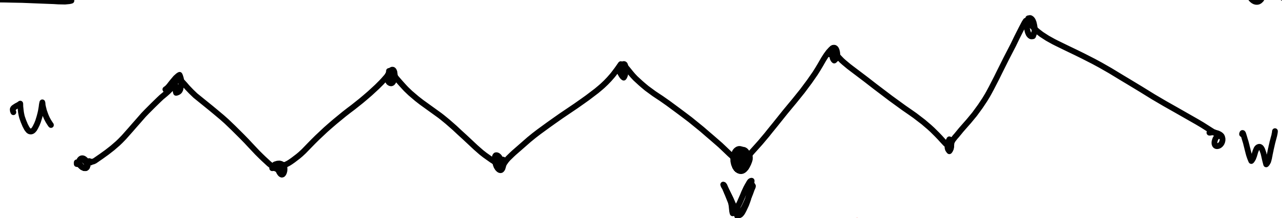
Same argument for edges bought by head. $\square$

Lemma

If girth $\geq 12 \log n$ then (i) diameter of G is $< 6 \log n$ and (ii) G is a tree. (ii) follows from (i).

Proof

$\delta(u,w)$ $6 \log n$



$d =$ depth

b: # neutral edges on path u to v.

SHOW: # bdry descendants

$$N(d,b) \geq n\, 2^{b - \frac{1}{2} d}$$

Get contradiction u has $\geq N(0,0) = n$ bdry descendants.

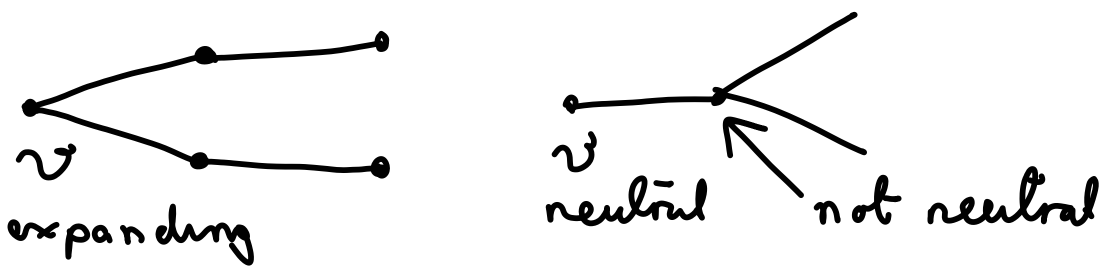
Backwards induction on b, d.

$$\underbrace{N(6\log n, b)}_{\geq 1} \geq n\,2^{b-3\log n} = \underbrace{2^b/n^2}_{\leq 1}$$

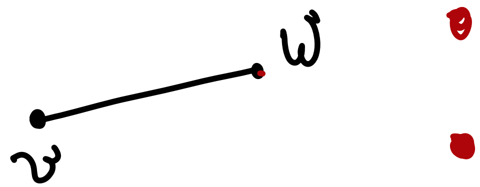
BASE CASE

$$N(d, 2\log n) \geq n^3 2^{-\frac{1}{2}d}$$

Back induction on d. $d = 6\log n$ ✓



$$N(d, 2\log n) \geq N(d+2, 2\log n) + N(d+2, 2\log n)$$ ✓



E & E   E & N   N & E

$$N(d,b) \geq N(d+2,b) + N(d+2,b)$$ ✓

N & N

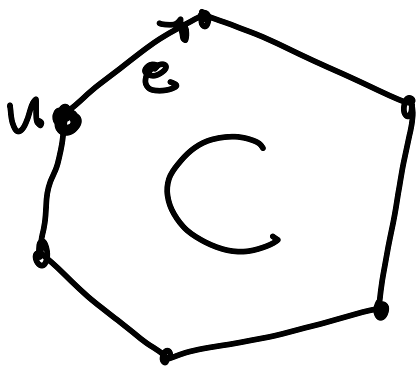
$$N(d,b) \geq N(d+2,b+1)$$ ✓

$\square$

Lemma

G is equilibrium and $\alpha > dn \log n$.
Then girth $\geq d \log n$.

Proof



u buys e.

Gain $\leq -\alpha + |C|(n-1)$

< 0

if $|C| < d \log n$.

$\square$

Proof of Thm

If $\alpha \geq 12 n \log n$ then girth $\geq 12 \log n$ and then G is a tree of depth $\leq 6 \log n$.

$$\Pi_2 \leq \frac{\alpha(n-1) + 6n^2 \log n}{\alpha(n-1) + 2(n-1)^2} \leq 1 + \frac{6n \log n}{\alpha}.$$

Cost of star

$\square$

Thm

$$\alpha \geq 1 \Rightarrow \Pi_\alpha \leq 15\left(1 + \min\left\{\frac{\alpha^3}{n}, \frac{n^3}{\alpha}\right\}^{1/3}\right).$$

Proof

$\text{Cost}(v)$ = cost to v.

$$\text{Dist}(v) = \sum_{w \neq v} \delta(v, w)$$

Fix u.

E_v = tree edges built by v in $T(w)$.

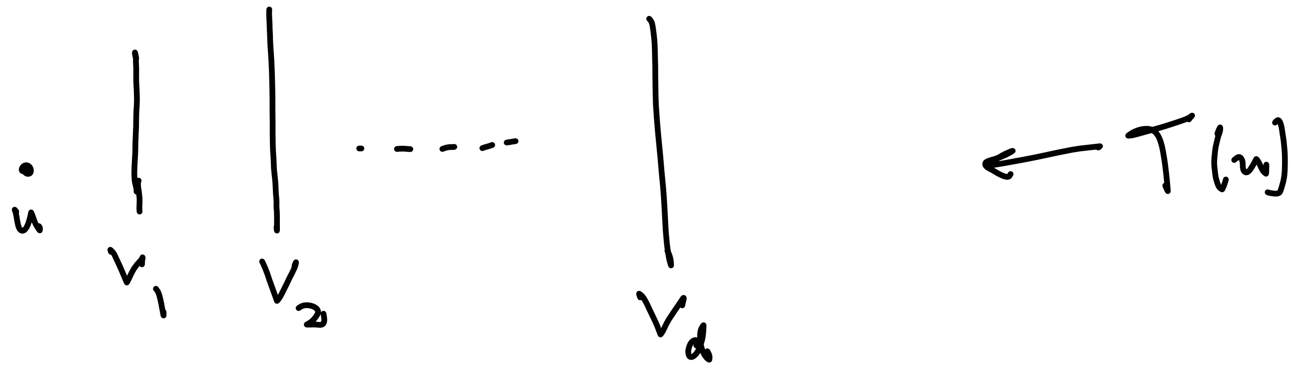
$$Cost(v) \leq \alpha(|E_v|+1) + Dist(v_0) + n - 1.$$

v discards $\underbrace{\text{non-tree edges}}_{X_v}$, and adds edge (v, v_0)

$$\underbrace{\alpha|E_v| + \alpha|X_v| + Dist(v)}_{Cost(v)} \leq \alpha|E_v| + \alpha + \sum_{w \neq u} (1 + \delta(u,w))$$

Summing gives

$$Cost(V) \leq 2\alpha(n-1) + n\,Dist(u) + (n-1)^2$$



(i) $d \le 9$

$\text{Dist}(u) \le 9n$.

$$\pi_\alpha \le \frac{2\alpha(n-1) + 10n^2}{\alpha(n-1) + n(n-1)}.$$

(ii) $d \ge 10$.

Write $\alpha = n^{3c-1}$ where $\frac{1}{3} \le c \le 1$.

$$V' = V_1 \cup V_2 \cup \cdots \cup V_r \qquad r = \left\lfloor \tfrac{2}{5} d \right\rfloor.$$

<u>Case 1</u> : $|V'| \geq \frac{2}{3} n^c$

Choose $w_0 \in V_d$. $\delta(w_0, V') \geq \left\lceil \frac{3}{5} d \right\rceil$.

No edge $(w_0, u) \Rightarrow$

$$\alpha > |V'| \left(\left\lceil \tfrac{3}{5} d \right\rceil - \left\lfloor \tfrac{2}{5} d \right\rfloor - 1 \right) \geq \frac{2}{3} n^c \frac{d}{10}$$

distance if you added edge

$$\Rightarrow \qquad d \leq \frac{15\alpha}{n^c}.$$

Case 2: $|V'| < \frac{2}{3}n^c$.

$\exists \lfloor \frac{1}{5}d \rfloor + 1 \le i_0 \le \lfloor \frac{2}{5}d \rfloor$ s.t. $|V_{i_0}| < \frac{2}{3}n^c / \lfloor \frac{1}{5}d \rfloor$.

$\exists v_0 \in V_{i_0}$ with at least

$$\frac{|V \setminus V'|}{|V_{i_0}|} \geq \frac{n/3}{\frac{2}{3}n^c / \lfloor \frac{1}{5}d \rfloor} \geq \frac{d}{20} n^{1-c}$$ descendents.

No (u, v_0) edge $\Rightarrow$

$$\alpha \geq \lfloor \frac{1}{5}d \rfloor \frac{d}{20} n^{1-c} \geq \frac{d^2}{100} n^{1-c}$$

distance reduction in u to $V \setminus V'$ if edge is built.

$$\Rightarrow \quad d \le 15 \sqrt{\frac{\alpha}{n^{1-c}}}$$

$$\Pi_\alpha \le \frac{2\alpha(n-1) + 15\alpha n^{2-c} + n^2}{\alpha(n-1) + n^2}$$

$$\le 3 + \frac{15\alpha n^{2-c}}{\alpha(n-1) + n^2}$$

$\alpha \le n$: $\Pi_\alpha \le 3 + \frac{15\alpha}{n^c} < 15\left(1 + \left(\alpha^2/n\right)^{\frac{1}{3}}\right)$

$\alpha > n$: $\Pi_\alpha \le 3 + 15 n^{1-c} = 15\left(1 + \left(n^2/\alpha\right)^{\frac{1}{3}}\right)$.

$\square$