Department of Mathematics
Carnegie Mellon University
21-484 Applied Graph Theory
Test 2

Name:______________________________

Problem	Points	Score
1	34	
2	33	
3	33	
Total	100	

Q1 (34 points)
Let $k = \chi(G)$ and let $C_1, C_2, \ldots, C_k$ be a partition of $V(G)$ into colour classes in some proper k-colouring.
(i) Show that for each $i \neq j$, G contains an edge $\{v, w\}$ with $v \in C_i, w \in C_j$.
(ii) Deduce the inequality

$$\epsilon(G) \geq \binom{\chi(G)}{2}.$$

Q2 (33 points)
A bipartite graph G has bipartition X, Y where $|X| = |Y| = n$. Let $N(S)$ denote the set of neighbours in Y of a subset $S \subseteq X$. Suppose that G has at least $(k-1)n + 1$ edges for integer $k \geq 1$. Show
(i) $|N(S)| \geq |S| - (n - k)$ for all $S \subseteq X$.
(ii) Deduce that G has a matching of size at least k.
[Hint: add a set Z of $n - k$ vertices to Y and join each vertex in Z to each vertex in X. Show that the new graph obtained has a matching of X into $Y \cup Z$.]

Q3 (33 points)
Let $\mathcal{X}$ denote a collection $X_1, X_2, \ldots X_n$ of k-subsets of X. $\mathcal{X}$ is 2-colourable if there is a Red-Blue colouring of the elements of the set X such that no X_i is mono-coloured i.e. has all of its elements coloured all Red or all Blue.
Show that if $n < 2^{k-1}$ then $\mathcal{X}$ is 2-colourable.
[Hint: colour X randomly and compute the expected number of sets which are mono-coloured.]