Department of Mathematics
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Discrete Mathematics 21-228

Answers to homework 7.

Q1: (a) For such partitions there are

(i) b_{n-1} where n occurs in a singleton set $\{n\}$.

(ii) $(n-1)b_{n-2}$ where n occurs in a doubleton $\{k, n\}$ i.e. $n-1$ choices for k and b_{n-2} choices for each k.

(b)

$$\begin{aligned} \frac{b_n}{(n-1)!} &= \frac{b_{n-1}}{(n-1)!} + \frac{b_{n-2}}{(n-2)!} \\ \sum_{n=2}^{\infty} \frac{b_n x^{n-1}}{(n-1)!} &= \frac{b_{n-1}x^{n-1}}{(n-1)!} + \frac{b_{n-2}x^{n-1}}{(n-2)!} \end{aligned} \tag{1}$$

But

$$b'(x) = \sum_{n=1}^{\infty} \frac{b_n x^{n-1}}{(n-1)!}$$

and so (1) is

$$b'(x) - 1 = b(x) - 1 + xb(x).$$

Q2: Let x_i denote the number of times colour i is used. We want the size of A where

$$A = \{x: \ x_1 + x_2 + \cdots + x_n = n, \ 0 \leq x_i \leq 3], \ 1 \leq i \leq n\}.$$

Let

$$A_i = \{x: \ x_1 + x_2 + \cdots + x_n = n, \ x_i \geq 4\}$$

and then

$$\begin{aligned} |A_I| &= |\{x: \ x_1 + x_2 + \cdots + x_n = n, \ x_i \geq 4, \ i \in I\}| \\ &= |\{x': \ x'_1 + x'_2 + \cdots + x'_n = n - 4|I|, \ x'_i \geq 0\}| \end{aligned}$$

and so

$$|A_I| = \binom{2n - 4|I| - 1}{n - 4|I|}.$$

So number of colourings is

$$\sum_{S\subseteq[n]}(-1)^{|S|}\binom{2n-4|I|-1}{n-4|I|}=\sum_{k=0}^{\lfloor n/4\rfloor}(-1)^k\binom{n}{k}\binom{2n-4k-1}{n-4k}.$$

Q3: Let $A_i=\{(\pi_1,\pi_2):\ \pi_1(i)=\pi_2(i)=i\}$ so that

$$A=\bigcap_{i=1}^{n}\bar{A}_i$$

is the set of pairs of permutations in question.
Now

$$|A_I|=(n-|I|)!^2$$

since we have fixed $|I|$ entries of both prmutations in defining A_I. Thus,

$$\begin{aligned}|A| &= \sum_{S\subseteq[n]}(-1)^{|S|}(n-|I|)!^2\\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)!^2.\end{aligned}$$