Department of Mathematics
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Discrete Mathematics 21-228

Answers to homework 6.
Q1: Let

$$\begin{aligned} P_n &= \{\pi : \text{there are no cycles of length 2}\} \\ P_n^{(k)} &= \{\pi \in P_n : n \text{ lies on a cycle of length k}\}, \qquad k = 1, 3, 4, \dots, n-1. \end{aligned}$$

If $\pi \in P_n^{(k)}$, remove the cycle containing n. This leaves a permutation in P_{n-k}. There are $(n-1)_{k-1} = (n-1)(n-2)\cdots(n-k+1)$ choices for the cycle containing n. Thus

$$|P_n^{(k)}| = (n-1)_{k-1} p_{n-k}$$

and

$$p_n = \sum_{\substack{k=1 \\ k \neq 2}}^{n-1} (n-1)_{k-1} p_{n-k}.$$

Q2:

$$B(n,k) = \sum_{\ell=0}^{n} \binom{n-1}{2\ell} B(n-2\ell-1, k-1).$$

$\binom{n-1}{2\ell} B(n-2\ell, k-1)$ counts the number of partitions in which n belongs to a subset of size $2\ell+1$. We choose the 2ℓ elements to go with n and then there are $B(n-2\ell-1, k-1)$ ways of partitioning the remainder. (Note that $B(m,s) = 0$ for $s > m$.
Q3:

$$\begin{aligned} a_n - 7a_{n-1} + 12a_{n-2} &= n(n-1)2^n \qquad\qquad n \geq 2 \\ \sum_{n=2}^{\infty} (a_n - 7a_{n-1} + 12a_n)x^n &= \sum_{n=2}^{\infty} n(n-1)2^n x^n \\ a(x) - (1+x) - 7x(a(x)-1) + 12x^2 a(x) &= \frac{8x^2}{(1-2x)^3} \\ a(x)(1 - 7x + 12x^2) &= \frac{8x^2}{(1-2x)^3} + 1 - 6x \end{aligned}$$

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\begin{aligned}
a(x) &= \frac{8x^2}{(1-2x)^3(1-3x)(1-4x)} + \frac{1-6x}{(1-3x)(1-4x)} \\
&= \frac{4}{(1-2x)^3} + \frac{12}{(1-2x)^2} + \frac{40}{1-2x} - \frac{69}{1-3x} + \frac{14}{1-4x} \\
&= \sum_{n=0}^{\infty} 4\binom{n+2}{2} 2^n x^n + 12\sum_{n=0}^{\infty} (n+1) 2^n x^n + \\
&\qquad\qquad 40\sum_{n=0}^{\infty} 2^n x^n - 69\sum_{n=0}^{\infty} 3^n x^n + 14\sum_{n=0}^{\infty} 4^n x^n.
\end{aligned}
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So

$$
a_n = \left(4\binom{n+2}{2} + 12n + 52\right) 2^n - 69 \times 3^n + 14 \times 4^n \qquad \text{for } n \geq 0.
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