Department of Mathematics
CARNEGIE MELLON UNIVERSITY

Discrete Mathematics 21-228

Answers to homework 5.

Q1 Let Z be the random variable which counts the relevant number of steps. Let $p_k = \mathbf{P}(Z = k)$ so that $p_k = 0$ unless $k = 2\ell+1$ for $1 \leq \ell < n$ or $k = 2n$. Next note that the number ν_ℓ of walks of length 2ℓ is given by

$$\nu_\ell = \sum_{r=\max\{0,2\ell-n\}}^{\min\{n,2\ell\}} \binom{2\ell}{r}.$$

Here r will be the number of FORWARD moves.
Now

$$p_{2\ell+1} = \frac{\frac{1}{\ell+1}\binom{2\ell}{\ell}}{\nu_\ell} \cdot \frac{1}{2} = \frac{\binom{2\ell}{\ell}}{2\nu_\ell(\ell+1)}. \tag{1}$$

Explanation: The clowns walk can be mapped to a grid walk where UP=FORWARD and RIGHT=BACK. Then the clown falls in after exactly $2\ell+1$ steps if the walk initially goes from $(0,0)$ to (ℓ,ℓ), stays above the diagonal for these 2ℓ steps, and then immediately goes below the diagonal. The fraction $\frac{\frac{1}{\ell+1}\binom{2\ell}{\ell}}{\nu_\ell}$ in (1) is the proportion of the the walks of length 2ℓ that go from $(0,0)$ to (ℓ,ℓ) without dropping below the diagonal and the $\frac{1}{2}$ is the probability that the next move goes below the diagonal.
Thus

$$\mathbf{E}(Z) = \sum_{\ell=1}^{n-1} \frac{(2\ell+1)\binom{2\ell}{\ell}}{2\nu_\ell(\ell+1)} + \frac{2n}{n+1}$$

Q2: The recurrence has characteristic equation

$$x^2 - 7x + 12 = 0.$$

This has roots 3,4 and so we try

$$a_n = c_1 3^n + c_2 4^n.$$

The equations for c_1, c_2 are

$$\begin{aligned} c_1 + c_2 &= 1 \qquad (n = 0) \\ 3c_1 + 4c_2 &= 1 \qquad (n = 1) \end{aligned}$$

So $c_1 = 3, c_2 = -2$ and

$$a_n = 3^{n+1} - 2^{2n+1}.$$

Q3: Let $C_n =$\{sequences\}. Then

$$C_n = C_n^{(0)} \cup C_n^{(100)} \cup C_n^{(1100)} \cup \cdots$$

where

$$\begin{aligned} C_n^{(0)} &= \{x \in C_n : \ x_1 = 1\} \\ C_n^{(100)} &= \{x \in C_n : \ x_1x_2x_3 = 100\} \\ C_n^{(1100)} &= \{x \in C_n : \ x_1x_2x_3x_4 = 1100\} \end{aligned}$$

etc.

$$|C_n^{(0)}| = c_{n-1},\ |C_n^{(100)}| = c_{n-3},\ |C_n^{(1100)}| = c_{n-4}, \ldots$$

The sequence in the recurrence ends with $c_0 + 2$. The c_0 counts those sequences which start with $n-2$ 1's followed by a 0 and the 2 counts the sequences which start with $n-3$ 1's.