Department of Mathematics
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Discrete Mathematics 21-228

Answers to homework 4.

Q1 Each time the person reaches for a ball, the probability of choosing a red ball is the same, $p = \frac{a}{a+b}$. These choices are made independently and so the number of choices until a red ball is chosen is distributed as a geometric random varaible with parameter p. Hence the expected number of times until a red ball is chosen is $1/p$.

Q2 This time the probability of finding a red ball increases. Let C denote the colour of the first ball chosen and let Z be the number of trials until a red ball is chosen. Then

$$\begin{aligned} \mathrm{E}(Z) &= \mathrm{E}(Z \mid C = \text{red}) \Pr(C = \text{red}) + \mathrm{E}(Z \mid C = \text{blue}) \Pr(C = \text{blue}) \\ &= 1 \times \frac{a}{a+b} + (1 + f(a, b-1)) \times \frac{b}{a+b} \\ &= 1 + \frac{b}{a+b} f(a, b-1). \end{aligned}$$

Now $f(a', b') = \frac{a'+b'+1}{a'+1}$ for $a' + b' \leq 2$ (assuming $a' \geq 1$) and so assume $f(a', b') = \frac{a'+b'+1}{a'+1}$ whenever $a' + b' \leq k-1$ and suppose that $a + b = k$. Then

$$f(a, b) = 1 + \frac{b}{a+b} f(a, b-1) = 1 + \frac{b}{a+b} \frac{a+b}{a+1} = \frac{a+b+1}{a+1}.$$

Q3: We define the following events:

$$\begin{aligned} \mathcal{A} &= \{\text{The cherry chosen after the pig eats, has a stone}\} \\ \mathcal{B} &= \{\text{The pig ate at least one stone}\} \\ \mathcal{B}_k &= \{\text{The pig ate exactly } k \text{ stones}\} \end{aligned}$$

Now

$$
\begin{aligned}
\Pr(\mathcal{A}) &= \sum_{k=0}^{n-m} \Pr(\mathcal{A} \mid \mathcal{B}_k) \Pr(\mathcal{B}_k) \\
&= \sum_{k=0}^{n-m} \frac{n-m-k}{n-p} \times \frac{\binom{n-m}{k}\binom{m}{p-k}}{\binom{n}{p}} \\
&= \sum_{k=0}^{n-m} \frac{n-m}{n-p} \times \frac{\binom{n-m-1}{k}\binom{m}{p-k}}{\binom{n}{p}} \qquad (1) \\
&= \frac{n-m}{n-p} \times \frac{\binom{n-1}{p}}{\binom{n}{p}} \\
&= \frac{n-m}{n}. \qquad (2)
\end{aligned}
$$

The final line is of course "obvious". The cherry alluded to in (a) is equally likely to be any cherry.
Now

$$
\begin{aligned}
\Pr(\mathcal{B} \mid \mathcal{A}) &= 1 - \Pr(\mathcal{B}_0 \mid \mathcal{A}) \\
&= 1 - \frac{\Pr(\mathcal{B}_0 \cap \mathcal{A})}{\Pr(\mathcal{A})} \\
&= 1 - \frac{\Pr(\mathcal{A} \mid \mathcal{B}_0) \Pr(\mathcal{B}_0)}{\Pr(\mathcal{A})} \\
&= 1 - \frac{n-m}{n-p} \times \frac{\binom{m}{p}}{\binom{n}{p}} \times \frac{n}{n-m} \qquad \text{using (1),(2)} \\
&= 1 - \frac{\binom{m}{p}}{\binom{n-1}{p}}
\end{aligned}
$$

The last line is again "obvious". Fix a cherry C with a stone. The probability that the pig eats no cherries with a stone, given that C is the cherry alluded to in (a), is precisely $\frac{\binom{m}{p}}{\binom{n-1}{p}}$.
The moral of this question is that sometimes a little thought can save a lot of calculation, but it nice to do the calculations correctly.