Department of Mathematics
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Discrete Mathematics 21-228

Answers to homework 3.

Q1 The number of ways of numbering the balls is 3^n, each way being equally likely. The number of ways of numbering the balls so that k are numbered 1 and k are numbered 2 is

$$\binom{n}{k, k, n-2k} = \binom{n}{k}\binom{n-k}{k}.$$

So the probability that there are exactly as many balls numbered with 1 as there are balls numbered with 2 is

$$3^{-n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k, k, n-2k}.$$

Q2: The number of outcomes for rolling n dice is 6^{6n}, each way being equally likely. The number of outcomes with n 1's and n 2's and ... n 6's is

$$\binom{6n}{n, n, n, n, n, n} = \binom{6n}{n}\binom{5n}{n}\binom{4n}{n}\binom{3n}{n}\binom{2n}{n}.$$

So the probability is

$$6^{-6n}\binom{6n}{n, n, n, n, n, n}.$$

Q3: The probability is 1/2. Let $\mathcal{P}_T$ denote the set of walks that go through $(n, n-1)$ and let $\mathcal{P}_R$ denote the rest of walks (that go through $(n-1, n)$. If P is a walk from $(0,0)$ to (n, n) then let $r(P)$ be that obtained by reflecting P in the diagonal i.e by interchanging R and U in the string representation of the path. If $P \in \mathcal{P}_T$ then $r(P) \in \mathcal{P}_R$ and vice-versa and $r(r(P)) = P$. Thus r is a bijection between $\mathcal{P}_T$ and $\mathcal{P}_R$. So $|\mathcal{P}_T| = |\mathcal{P}_R|$ implying the probability 1/2.

Answer to the optional pigeon-hole question.

By Erdős-Szekerés, the sequence $a_1, a_2, \ldots, a_m$ contains a monotone sequence $a_{j_1}, a_{j_2}, \ldots, a_{j_{n^2+1}}$ which is monotone ($m = (n^2)^2 + 1$). By Erdős-Szekerés again, the sequence $b_{j_1}, b_{j_2}, \ldots, b_{j_{n^2+1}}$ contains a monotone subsequence $b_{i_1}, b_{i_2}, \ldots, b_{i_{n+1}}$. Since $a_{i_1}, a_{i_2}, \ldots, a_{i_{n+1}}$ is also monotone (it is a subsequence of $a_{j_1}, a_{j_2}, \ldots, a_{j_{n^2+1}}$) we are done.