Department of Mathematics
CARNEGIE MELLON UNIVERSITY

Discrete Mathematics 21-228

Answers to homework 2.
Q1 By induction on k for every n.
Base case: $k = 0$, trivial.
Assume true for k. Then

$$\begin{aligned}\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^k\binom{n}{k} + (-1)^{k+1}\binom{n}{k+1} &= (-1)^k\binom{n-1}{k} + (-1)^{k+1}\binom{n}{k+1} \\ &= (-1)^{k+1}\left(\binom{n}{k+1} - \binom{n}{k}\right) \\ &= (-1)^{k+1}\binom{n-1}{k+1}.\end{aligned}$$

Q2 Using the binomial theorem,

$$\begin{aligned}(1+i)^n &= \binom{n}{0} + \binom{n}{1}i + \binom{n}{2}i^2 + \binom{n}{3}i^3 + \binom{n}{4}i^4 + \cdots + \\ &= \left(\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots\right) + i\left(\binom{n}{1} - \binom{n}{3} + \cdots\right)\end{aligned}$$

Thus

$$Re\ (1+i)^n = \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots.$$

On the other hand

$$(1+i)^n = 2^{n/2}(\cos\pi/4 + i\sin\pi/4)^n = 2^{n/2}(\cos n\pi/4 + i\sin n\pi/4)$$

and so

$$Re\ (1+i)^n = 2^{n/2}\cos n\pi/4.$$

Q3 The number of ways of choosing two k-sets A, B and then choosing $a \in A, b \in B$ is $\binom{n}{k}^2 k^2$. Sum over k to get the total number of ways of choosing A, B, a, b.
On the other hand there are n^2 ways of choosing a, b and having a, b we must choose A containing a and B containing b. Let $X = [n] \setminus \{a\}$ and $Y = [n] \setminus \{b\}$. We must now count the number of ways of choosing two sets $A' \subseteq X, B' \subseteq Y$ of equal size. This is $\binom{2n-2}{n-1}$ giving the result we want. To see this, replace X by $\{x_i : i \in X\}$ and Y by $\{y_i : i \in Y\}$. A choice of ℓ elements L_X of X and ℓ elements L_Y of Y is associated with the set $L_X \cup (Y \setminus L_Y)$ which has size $n-1$. Conversely,

any $n-1$ subset of $X \cup Y$ contains ℓ elements L'_X of X and $n-1-\ell$ elements L'_Y of Y. L'_X defines A' and L'_Y defines $X \setminus B'$.

Here is an algebraic version of the above combinatorial argument from DK. Using Vandermonde's identity,

$$\begin{aligned} n^2\binom{2n-2}{n-1} &= n^2\sum_{i=0}^{n-1}\binom{n-1}{i}\binom{n-1}{n-1-i} = n^2\sum_{i=0}^{n-1}\binom{n-1}{i}\binom{n-1}{i} \\ &= \sum_{i=1}^{n} i^2\binom{n}{i}^2 \end{aligned}$$

since

$$n\binom{n-1}{i} = (i+1)\binom{n}{i+1}$$

as can easily be verified from the factorial expression for binomial coefficeints.