Department of Mathematics
CARNEGIE MELLON UNIVERSITY

Discrete Mathematics 21-228

Answers to homework 10.

Q1: We prove this by induction on k. C_2 is a cycle of length 4 and so trivially has a Hamilton cycle. Now suppose $k > 3$ and assume that C_{k-1} is Hamiltonian.

Partition the vertices V_k of C_k into two sets $V_{k,j} = \{\mathbf{x} \in V_k : \; x_k = j\}, j = 0, 1$ and let $\tau : V_{k,0} \to V_{k,1}$ be defined by $\tau(x_1, x_2, \dots, x_{k-1}, 0) = (x_1, x_2, \dots, x_{k-1}, 1)$.

The subgraph $C_{k,0}$ of C_k induced by $V_{k,0}$ is isomorphic to C_{k-1} and so is Hamiltonian. Let $H_0 = (\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(2^{k-1})}, \mathbf{x}^{(1)})$ be a Hamilton cycle in $C_{k,0}$. Let $H_1 = (\mathbf{y}^{(1)} = \tau(\mathbf{x}^{(1)}), \mathbf{y}^{(2)} = \tau(\mathbf{x}^{(2)}), \dots, \mathbf{y}^{(2^{k-1})} = \tau(\mathbf{x}^{(2^{k-1})}), \mathbf{x}^{(1)})$. This is a Hamilton cycle in $C_{k,1}$ and then

$$H = (\mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \dots \mathbf{x}^{(2^{k-1})}, \mathbf{x}^{(1)}, \mathbf{y}^{(1)}, \mathbf{y}^{(2^{k-1})}, \mathbf{y}^{(2^{k-1}-1)}, \dots, \mathbf{y}^{(2)}, \mathbf{x}^{(2)})$$

is a Hamilton cycle in C_k.

Q2: Add k new edges $e_1, e_2, \dots, e_k$ joining up the $2k$ vertices of odd degree. The new graph is Eulerian and has an Euler tour T. Deleting $e_1, e_2, \dots, e_k$ from T yields the k desired paths.

Q3: Let P be a Hamilton path of G. Removing $s = |S|$ vertices from P breaks the path into $\leq s + 1$ sub-paths. Each of these sub-paths will be part of a component of $G \setminus S$ and so $G \setminus S$ has at most $s + 1$ components itself.