Department of Mathematics
Carnegie Mellon University

21-228 Discrete Mathematics: Section 1
Test 2

Name:______________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
Two positions (i_1, j_1), (i_2, j_2) are *adjacent* if $|i_1 - i_2| + |j_1 - j_2| = 1$ i.e. they are in same row (resp. column) and differ by one in column (resp. row) number.
Let b_n be the number of $2 \times n$ 0/1 matrices with no pair of 1's in adjacent positions.

Counted

1	0	1	0	1
0	1	0	0	0

Not Counted

1	0	1	0	1
0	1	1	0	0

Show that if $b_0 = 1, b_1 = 3$ then

$$b_n = b_{n-1} + 2(b_{n-2} + \cdots + b_0 + 1) \qquad n \geq 2.$$

DO NOT SOLVE THE EQUATION
[Hint: partition set of matrices according to first occurence of column $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$]

Q2: (33pts)
The sequence $u_n, n \geq 0$ satisfies $u_0 = 1$ and

$$u_n - 2u_{n-1} = n + 1, \qquad n \geq 1.$$

(i) Show that the generating function

$$u(x) = \sum_{n=0}^{\infty} u_n x^n = \frac{1}{(1-2x)(1-x)^2}.$$

(ii) Deduce from (i) that

$$u_n = 2^{n+2} - n - 3.$$

Q3: (34pts)
Let $A = \{f : [n] \to [n] \text{ such that } f(i) \neq i \text{ for all } i \in [n]\}$. Clearly $|A| = (n-1)^n$. On the other hand, the binomial theorem gives

$$(n-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k n^{n-k}. \tag{1}$$

Express A as $\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n$ for a suitable definition of the A_i and then use inclusion-exclusion to obtain the right hand side of (**??**).