**Department of Mathematics**
**Carnegie Mellon University**

21-228 Discrete Mathematics
Test 2

Name:________________________________________

| Problem | Points | Score |
|---|---|---|
| 1 | 33 | |
| 2 | 33 | |
| 3 | 34 | |
| Total | 100 | |

**Q1: (33pts)**
Solve the following linear recurrence:

$$u_n - 7u_{n-1} + 12u_{n-2} = 0, \qquad\qquad n \geq 2.$$

$u_0 = 2,\ u_1 = 7.$

**Q2: (33pts)**

**(a)** A box contains $a \geq 2$ red and $b$ blue balls. A person reaches into the box and removes two random balls. If either is blue he/she replaces them both and tries again. Show that the expected number of tries until both balls drawn are red is $\frac{(a+b)(a+b-1)}{a(a-1)}$.

**(b)** A box contains $a \geq 2$ red and $a$ blue balls. A person repeatedly reaches into the box, removes two random balls and discards them. Show that the probability that the person never in the course of $a$ attempts chooses two balls of the same colour at one time is $\frac{2^a}{\binom{2a}{a}}$.

**Q3: (34pts)**

**(a)** Let $A, B, C$ be randomly chosen subsets of $[n] = \{1, 2, \dots, n\}$. Prove that

$$\mathbf{P}(A \cap B \subseteq C) = \left(\frac{7}{8}\right)^n.$$

**(b)** Let $m = \lfloor (8/7)^{n/3} \rfloor$. Use (a) and Boole's inequality to prove the existence of $A_1, A_2, \dots, A_m \subseteq [n]$ such that for all distinct $i, j, k$,

$$A_i \cap A_j \nsubseteq A_k.$$

(Hint: choose $A_1, A_2, \dots, A_m$ randomly.)