Department of Mathematics
Carnegie Mellon University

21-228 Discrete Mathematics
Answers to Test 1

Q1 Let x_i denote the number of times that colour i is used. If $x_i' = x_i - i$ then $x_i' \geq 0$ and

$$\sum_{i=1}^{n} x_i' = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} i = m - \frac{n(n+1)}{2}. \tag{1}$$

The number of colourings is then the number of solutions to (1) which is

$$\binom{m - \frac{n(n+1)}{2} + n - 1}{n-1} = \binom{m - \frac{n^2-n+2}{2}}{n-1}.$$

Q2 Place i into box $S_i \pmod{n}$ for $i = 0, 1, 2, \ldots, n$. We have placed $n+1$ items into n boxes. By the pigeon-hole principle, there is a box containing at least 2 indices, i, j say. This proves (i).
If $S_i \equiv S_j \pmod{n}$ then

$$S_j - S_i = a_{i+1} + \cdots + a_j \equiv 0 \pmod{n}.$$

Let $k = i + 1$ and $l = j$.
Q3 There are $\binom{2n}{n}$ walks from $(0,0)$ to (n,n) of which $\frac{1}{n+1}\binom{2n}{n}$ do not go below the diagonal. Thus the probability is

$$\frac{\frac{1}{n+1}\binom{2n}{n}}{\binom{2n}{n}} = \frac{1}{n+1}.$$