Discrete Probability

\( \Omega \) is a finite or countable set – called the *Probability Space*

\[ P : \Omega \to \mathbb{R}^+ . \]

\[ \sum_{\omega \in \Omega} P(\omega) = 1 . \]

If \( \omega \in \Omega \) then \( P(\omega) \) is the *probability* of \( \omega \).
Fair Coin
\[ \Omega = \{H, T\}, \ P(H) = P(T) = 1/2. \]

Dice
\[ \Omega = \{1, 2, \ldots, 6\}, \ P(i) = 1/6, \ 1 \leq i \leq 6. \]

Both are examples of a uniform distribution:
\[ P(\omega) = \frac{1}{|\Omega|} \quad \forall \omega \in \Omega. \]
Geometric or number of Bernouilli trials until success
\[ \Omega = \{1, 2, \ldots, \}, \ P(k) = (1 - p)^{k-1}p, \quad k \in \Omega. \]

Repeat ”experiment” until success – \( k \) is the total number of trials.
\( p \) is the probability of success.

\[ P(S) = p, \ P(FS) = p(1 - p), \]
\[ P(FFS) = p^2(1 - p), \ P(FFFS) = (1 - p)^3p \ldots, \]

Note that
\[ \sum_{k=1}^{\infty} (1 - p)^{k-1}p = \frac{p}{1 - (1 - p)} = 1. \]
Roll Two Dice

**Probability Space 1:**
\[ \Omega = [6]^2 = \{(x_1, x_2) : 1 \leq x_1, x_2 \leq 6\} \]
\[ P(x_1, x_2) = 1/36 \text{ for all } x_1, x_2. \]

**Probability Space 2:**
\[ \Omega = \{2, 3, 4, \ldots, 12\} \]
\[ P(2) = 1/36, P(3) = 2/36, P(4) = 3/36, \]
\[ \ldots, P(12) = 1/36. \]
Events

$A \subseteq \Omega$ is called an event.

$$P(A) = \sum_{\omega \in A} P(\omega).$$

(i) Two Dice

$A = \{x_1 + x_2 = 7\}$

where $x_i$ is the value of dice $i$.

$A = \{(1,6),(2,5),\ldots,(6,1)\}$ and so

$$P(A) = \frac{1}{6}.$$
(ii) **Pennsylvania Lottery**

Choose 7 numbers $I$ from $[80]$. The state randomly chooses $J \subseteq [80], |J| = 11$.

\[ WIN = \{ J : J \supseteq I \}. \]

$\Omega = \{11 \text{ element subsets of } [80]\}$ with uniform distribution.

$|WIN| = \text{no. subsets which contain } I - \binom{73}{4}$.

\[ P(WIN) = \frac{\binom{73}{4}}{\binom{80}{11}} = \frac{\binom{11}{7}}{\binom{80}{7}} \]

\[ = \frac{9}{86637720} \approx \frac{1}{9,626,413}. \]
Poker
Choose 5 cards at random. \(|\Omega| = \binom{52}{5}\), uniform distribution.

(i) **Triple** – 3 cards of same value e.g. \(Q,Q,Q,7,5\).
\[P(\text{Triple}) = \frac{13 \times 4 \times 48 \times 44}{2\binom{52}{5}} \approx .021.\]

(ii) **Full House** – triple plus pair e.g. \(J,J,J,7,7\).
\[P(\text{Full House}) = \frac{13 \times 4 \times 12 \times 6}{\binom{52}{5}} \approx .007.\]

(iii) **Four of kind** – e.g. \(9,9,9,9,J\).
\[P(\text{Four of Kind}) = \frac{13 \times 48}{\binom{52}{5}} = 1/16660.\]
Birthday Paradox

\[ \Omega = [n]^k - \text{uniform distribution, } |\Omega| = n^k. \]
\[ D = \{ \omega \in \Omega; \text{symbols are distinct}\}. \]
\[ P(D) = \frac{n(n-1)(n-2)\ldots(n-k+1)}{n^k}. \]

\[ n = 365, k = 26 \] – birthdays of 26 randomly chosen people.
\[ P(D) < .5 \] i.e. probability 26 randomly chosen people have distinct birthdays is < .5. (Assumes people are born on random days).
Reliability through redundancy: space ship has 7 independent on board computers.

The navigational decisions are reached by a majority vote of the seven computers.

If each computer is correct with probability \( p = .99 \), what is the probability the system gives a correct answer?

\[
\sum_{i=4}^{7} \binom{7}{i} p^i (1 - p)^{7-i} = .99999996583 \ldots .
\]