Ramsey’s Theorem

Suppose we 2-colour the edges $K_6$ of Red and Blue. There *must* be either a Red triangle or a Blue triangle.

This is not true for $K_5$. 
There are 3 edges of the same colour incident with vertex 1, say (1,2), (1,3), (1,4) are Red. Either (2,3,4) is a blue triangle or one of the edges of (2,3,4) is Red, say (2,3). But the latter implies (1,2,3) is a Red triangle.
Ramsey’s Theorem

For all positive integers $k, \ell$ there exists $R(k, \ell)$ such that if $N \geq R(k, \ell)$ and the edges of $K_N$ are coloured Red or Blue then either there is a “Red $k$-clique” or there is a “Blue $\ell$-clique.

A clique is a complete subgraph and it is Red if all of its edges are coloured red etc.

\[
\begin{align*}
R(1, k) &= R(k, 1) = 1 \\
R(2, k) &= R(k, 2) = k
\end{align*}
\]
Theorem 1

\[ R(k, \ell) \leq R(k, \ell - 1) + R(k - 1, \ell). \]

Proof Let \( N = R(k, \ell - 1) + R(k - 1, \ell) \).

\[ V_R = \{ (x : (1, x) \text{ is coloured Red}) \} \quad \text{and} \quad V_B = \{ (x : (1, x) \text{ is coloured Blue}) \}. \]
\[ |V_R| \geq R(k - 1, \ell) \text{ or } |V_B| \geq R(k, \ell - 1). \]

Since

\[ |V_R| + |V_B| = N - 1 \]
\[ = R(k, \ell - 1) + R(k - 1, \ell) - 1. \]

Suppose for example that \( |V_R| \geq R(k - 1, \ell) \). Then either \( V_R \) contains a Blue \( \ell \)-clique – done, or it contains a Red \( k - 1 \)-clique \( K \). But then \( K \cup \{1\} \) is a Red \( k \)-clique.

Similarly, if \( |V_B| \geq R(k, \ell - 1) \) then either \( V_B \) contains a Red \( k \)-clique – done, or it contains a Blue \( \ell - 1 \)-clique \( L \) and then \( L \cup \{1\} \) is a Blue \( \ell \)-clique. \( \square \)
Theorem 2

\[ R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}. \]

Proof: Induction on \( k + \ell \). True for \( k + \ell \leq 5 \) say. Then

\[
R(k, \ell) \leq R(k, \ell - 1) + R(k - 1, \ell) \\
\leq \binom{k + \ell - 3}{k - 1} + \binom{k + \ell - 3}{k - 2} \\
= \binom{k + \ell - 2}{k - 1}.
\]

So, for example,

\[
R(k, k) \leq \binom{2k - 2}{k - 1} \leq 4^k
\]
Theorem 3

\[ R(k, k) > 2^{k/2} \]

**Proof** We must prove that if \( n \leq 2^{k/2} \) then there exists a Red-Blue colouring of the edges of \( K_n \) which contains no Red \( k \)-clique and no Blue \( k \)-clique. We can assume \( k \geq 4 \) since we know \( R(3, 3) = 6 \).

We show that this is true with positive probability in a *random* Red-Blue colouring. So let \( \Omega \) be the set of all Red-Blue edge colourings of \( K_n \) with uniform distribution. Equivalently we independently colour each edge Red with probability 1/2 and Blue with probability 1/2.

Let

\( \mathcal{E}_R \) be the event: \{There is a Red \( k \)-clique\}

\( \mathcal{E}_B \) be the event: \{There is a Blue \( k \)-clique\}.

We show

\[ \Pr(\mathcal{E}_R \cup \mathcal{E}_B) < 1. \]
Let $C_1, C_2, \ldots, C_N$, $N = \binom{n}{k}$ be the vertices of the $N$ $k$-cliques of $K_n$. Let $\mathcal{E}_{R,j}$ be the event: \{ $C_j$ is Red \}. Now

\[
\begin{align*}
\Pr(\mathcal{E}_R \cup \mathcal{E}_B) & \leq \Pr(\mathcal{E}_R) + \Pr(\mathcal{E}_B) \\
& = 2\Pr(\mathcal{E}_R) \\
& = 2\Pr\left( \bigcup_{j=1}^{N} \mathcal{E}_{R,j} \right) \\
& \leq 2 \sum_{j=1}^{N} \Pr(\mathcal{E}_{R,j}) \\
& = 2 \sum_{j=1}^{N} \left( \frac{1}{2} \right)^{\binom{k}{2}} \\
& = 2 \frac{n^k}{k!} \left( \frac{1}{2} \right)^{\binom{k}{2}} \\
& \leq 2^{\frac{2k^2}{2}} \frac{k!}{k!} \left( \frac{1}{2} \right)^{\binom{k}{2}} \\
& = 2^{1+k/2} \\
& < 1.
\end{align*}
\]