Properties of binomial coefficients

• Symmetry

\[ \binom{n}{r} = \binom{n}{n-r} \]

Choosing \( r \) elements to include is equivalent to choosing \( n-r \) elements to exclude.

• Pascal’s Triangle

\[ \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \]

A \( k+1 \)-subset of \([n+1]\) either
(i) includes \( n+1 \) —— \( \binom{n}{k} \) choices or
(ii) does not include \( n+1 \) —— \( \binom{n}{k+1} \) choices.
Pascal’s Triangle

The following array of binomial coefficients, constitutes the famous triangle:

```
    1
   1 1
  1 2 1
 1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
   ...
```
Generalisation
\[ \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}. \] (1)

Proof 1: induction on \( n \) for arbitrary \( k \).

Base case: \( n = k \);

\[ \binom{k}{k} = \binom{k+1}{k+1} \]

Inductive Step: assume true for \( n \geq k \).

\[ \sum_{m=k}^{n+1} \binom{m}{k} = \sum_{m=k}^{n} \binom{m}{k} + \binom{n+1}{k} \]

\[ = \binom{n+1}{k+1} + \binom{n+1}{k} \quad \text{Induction} \]

\[ = \binom{n+2}{k+1}. \quad \text{Pascal’s triangle} \]
**Proof 2:** Combinatorial argument.

If $S$ denotes the set of $k + 1$-subsets of $[n + 1]$ and $S_m$ is the set of $k + 1$-subsets of $[n + 1]$ which have largest element $m + 1$ then

- $S_k, S_{k+1}, \ldots, S_n$ is a partition of $S$.

- $|S_k| + |S_{k+1}| + \cdots + |S_n| = |S|$.

- $|S_m| = \binom{m}{k}$.

This proves the result.
Vandermonde’s Identity

\[ \sum_{r=0}^{k} \binom{m}{r} \binom{n}{k-r} = \binom{m+n}{k}. \] (2)

Ex:

\[
\binom{6}{0}\binom{8}{4} + \binom{6}{1}\binom{8}{3} + \binom{6}{2}\binom{8}{2} + \binom{6}{3}\binom{8}{1} + \binom{6}{4}\binom{8}{0} = \binom{14}{4}
\]

Split \([m+n]\) into \(A = [m]\) and \(B = [m+n] \setminus [m]\). Let \(S\) denote the set of \(k\)-subsets of \([m+n]\) and let \(S_r = \{X \in S : |X \cap A| = r\}\). Then

- \(S_0, S_1, \ldots, S_m\) is a partition of \(S\).
- \(|S_0| + |S_1| + \cdots + |S_m| = |S|\).
- \(|S_r| = \binom{m}{r}\binom{n}{k-r}\).
- \(|S| = \binom{m+n}{k}\).

This proves the identity.
Binomial Theorem

\[(1 + x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r. \quad (3)\]

Coefficient \(x^r\) in \((1 + x)(1 + x) \cdots (1 + x)\): choose \(x\) from \(r\) brackets and 1 from the rest.

The proof of equation (3) assumed that \(n\) was an integer. The binomial theorem remains true for all real (or complex) \(n\) provided \(|x| \leq 1\) i.e.

\[(1 + x)^\alpha = \sum_{r=0}^{\infty} \binom{\alpha}{r} x^r\]

where \(\binom{\alpha}{r} = \alpha(\alpha - 1) \cdots (\alpha - r + 1)/r!\) – proof in any standard calculus text book.
Newton’s Binomial Theorem

\[(1 + x)^\alpha = \sum_{k=0}^{\infty} \frac{\alpha(\alpha - 1) \ldots (\alpha - k + 1)}{k!} x^k,\]

for real \(\alpha\) and \(|x| < 1\).

\[f(x) = (1 + x)^\alpha\]
\[f^{(k)}(x) = \alpha(\alpha - 1) \ldots (\alpha - k + 1)(1 + x)^{\alpha - k}.\]
\[f^{(k)}(0) = \alpha(\alpha - 1) \ldots \alpha - k + 1.\]

Taylor’s theorem

\[f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k\]
yields the theorem.
Example 1

\[(1 - x)^{-n} = \sum_{m=0}^{\infty} \frac{(-n)(-n-1)\ldots(-n-m+1)}{m!}(-x)^m\]

\[= \sum_{m=0}^{\infty} \frac{n(n+1)\ldots(n+m-1)}{m!}x^m\]

\[= \sum_{m=0}^{\infty} \binom{n+m-1}{m}x^m.\]

So if \(m = 3\) then

\[\frac{1}{(1 - x)^3} = \]

\[= \binom{2}{0} + \binom{3}{1}x + \binom{4}{2}x^2 + \ldots + \binom{n+2}{n}x^n + \ldots\]

\[= 1 + 3x + 6x^2 + \ldots + \frac{(n + 1)(n + 2)}{2}x^n + \ldots\]
(1 + x)^{1/2} = \\
= 1 + \sum_{k=1}^{\infty} \frac{(1/2)(1/2 - 1) \cdots (1/2 - k + 1)}{k!} x^k \\
= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} 1 \times 3 \times \cdots \times (2k - 3)}{2^k} \frac{1}{k!} x^k \\
= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (2k - 2)!}{2^k (2 \times 4 \times \cdots \times (2k - 2)) k!} x^k \\
= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (2k - 2)!}{2^k 2^{k-1} (k - 1)! k!} x^k \\
= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k 2^{2k-1}} \binom{2k-2}{k-1} x^k
Applications

• $x = 1$:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = (1 + 1)^n = 2^n.$$  
LHS counts the number of subsets of all sizes in $[n]$.

• $x = -1$:

$$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = (1 - 1)^n = 0,$$

i.e.

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

and number of subsets of even cardinality

$= \text{number of subsets of odd cardinality.}$
\[
\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.
\]

Differentiate both sides of the Binomial Theorem w.r.t. \(x\).

\[
(n(1 + x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}x^{k-1}.
\]

Now put \(x = 1\).