Trees

A tree is a graph which is

(a) Connected and

(b) has no cycles (acyclic).
Lemma 1  Let the components of $G$ be $C_1, C_2, \ldots, C_r$, Suppose $e = (u, v) \notin E, u \in C_i, v \in C_j$.

(a) $i = j \Rightarrow \omega(G + e) = \omega(G)$.

(b) $i \neq j \Rightarrow \omega(G + e) = \omega(G) - 1$. 
Proof Every path $P$ in $G + e$ which is not in $G$ must contain $e$. Also,

$$\omega(G + e) \leq \omega(G).$$

Suppose

$$(x = u_0, u_1, \ldots, u_k = u, u_{k+1} = v, \ldots, u_\ell = y)$$

is a path in $G + e$ that uses $e$. Then clearly $x \in C_i$ and $y \in C_j$.

(a) follows as now no new relations $x \sim y$ are added.

(b) Only possible new relations $x \sim y$ are for $x \in C_i$ and $y \in C_j$. But $u \sim v$ in $G + e$ and so $C_i \cup C_j$ becomes (only) new component. \qed
Lemma 2  \( G = (V, E) \) is acyclic (forest) with (tree) components \( C_1, C_2, \ldots, C_k \). \( |V| = n. \ e = (u, v) \notin E, \ u \in C_i, \ v \in C_j. \)

(a)  \( i = j \Rightarrow G + e \) contains a cycle.

(b)  \( i \neq j \Rightarrow G + e \) is acyclic and has one less component.

(c)  \( G \) has \( n - k \) edges.
(a) $u, v \in C_i$ implies there exists a path $(u = u_0, u_1, \ldots, u_\ell = v)$ in $G$.

So $G + e$ contains the cycle $u_0, u_1, \ldots, u_\ell, u_0$. 
(b) Suppose $G + e$ contains the cycle $C$. $e \in C$ else $C$ is a cycle of $G$.

$$C = (u = u_0, u_1, \ldots, u_\ell = v, u_0).$$

But then $G$ contains the path $(u_0, u_1, \ldots, u_\ell)$ from $u$ to $v$ – contradiction.

The drop in the number of components follows from Lemma 1.
The rest follows from
(c) Suppose \( E = \{e_1, e_2, \ldots, e_r\} \) and
\[ G_i = (V, \{e_1, e_2, \ldots, e_i\}) \] for \( 0 \leq i \leq r \).

**Claim:** \( G_i \) has \( n - i \) components.

Induction on \( i \).

\( i = 0 \): \( G_0 \) has no edges.

\( i > 0 \): \( G_{i-1} \) is acyclic and so is \( G_i \). It follows from part (a) that \( e_i \) joins vertices in distinct components of \( G_{i-1} \). It follows from (b) that \( G_i \) has one less component than \( G_{i-1} \).

**End of proof of claim**

Thus \( r = n - k \) (we assumed \( G \) had \( k \) components).  \( \square \)
Corollary 1  If a tree $T$ has $n$ vertices then

(a) It has $n - 1$ edges.

(b) It has at least 2 vertices of degree 1, ($n \geq 2$).

Proof   (a) is part (c) of previous lemma. $k = 1$ since $T$ is connected.

(b) Let $s$ be the number of vertices of degree 1 in $T$. There are no vertices of degree 0 – these would form separate components. Thus

$$2n - 2 = \sum_{v \in V} d_T(v) \geq 2(n - s) + s.$$ 

So $s \geq 2$.  

\[\square\]
Theorem 1 Suppose \(|V| = n\) and \(|E| = n - 1\). The following three statements become equivalent.

(a) \(G\) is connected.

(b) \(G\) is acyclic.

(c) \(G\) is a tree.

Let \(E = \{e_1, e_2, \ldots, e_{n-1}\}\) and \(G_i = (V, \{e_1, e_2, \ldots, e_i\})\) for \(0 \leq i \leq n - 1\).
(a) ⇒ (b): $G_0$ has $n$ components and $G_{n-1}$ has 1 component. Addition of each edge $e_i$ must reduce the number of components by 1 – Lemma 2(b). Thus $G_{i-1}$ acyclic implies $G_i$ is acyclic. (b) follows as $G_0$ is acyclic.

(b) ⇒ (c): We need to show that $G$ is connected. Since $G_{n-1}$ is acyclic, $\omega(G_i') = \omega(G_{i-1}') - 1$ for each $i$ – Lemma 2(b). Thus $\omega(G_{n-1}') = 1$.

(c) ⇒ (a): trivial.
**Corollary 2**  If $v$ is a vertex of degree 1 in a tree $T$ then $T - v$ is also a tree.

**Proof**  Suppose $T$ has $n$ vertices and $n - 1$ edges. Then $T - v$ has $n - 1$ vertices and $n - 2$ edges. It acyclic and so must be a tree.  \qed
Cut edges

$e$ is a cut edge of $G$ if $\omega(G - e) > \omega(G)$.

Theorem 2  

$e = (u, v)$ is a cut edge iff $e$ is not on any cycle of $G$.

Proof  

$\omega$ increases iff there exist $x \sim y \in V$ such that all walks from $x$ to $y$ use $e$.

Suppose there is a cycle $(u, P, v, u)$ containing $e$. Then if $W = x, W_1, u, v, W_2, y$ is a walk from $x$ to $y$ using $e$, $x, W_1, P, W_2, y$ is a walk from $x$ to $y$ that doesn’t use $e$. Thus $e$ is not a cut edge.
If $e$ is not a cut edge then $G - e$ contains a path $P$ from $u$ to $v$ ($u \sim v$ in $G$ and relations are maintained after deletion of $e$). So $(v, u, P, v)$ is a cycle containing $e$. 

\[\square\]

**Corollary 3** A connected graph is a tree iff every edge is a cut edge.
Corollary 4  Every finite connected graph $G$ contains a spanning tree.

Proof  Consider the following process: starting with $G$,

1. If there are no cycles – **stop**.

2. If there is a cycle, delete an edge of a cycle.

Observe that (i) the graph remains connected – we delete edges of cycles. (ii) the process must terminate as the number of edges is assumed finite.

On termination there are no cycles and so we have a connected acyclic spanning subgraph i.e. we have a spanning tree. \(\square\)
Theorem 3  Let $T$ be a spanning tree of $G = (V, E)$, $|V| = n$. Suppose $e = (u, v) \in E \setminus T$.

(a) $T + e$ contains a unique cycle $C(e)$.

(b) $f \in C(e)$ implies that $T + e - f$ is a spanning tree of $G$. 

\[ \begin{align*}
\text{(a)} &\quad T + e \text{ contains a unique cycle } C(e). \\
\text{(b)} &\quad f \in C(e) \implies T + e - f \text{ is a spanning tree of } G.
\end{align*} \]
Proof  (a) Lemma 2(a) implies that $T + e$ has a cycle $C$. Suppose that $T + e$ contains another cycle $C' \neq C$. Let edge $g \in C' \setminus C$. $T' = T + e - g$ is connected, has $n - 1$ edges. But $T'$ contains a cycle $C$, contradicting Lemma 2(b).

(b) $T + e - f$ is connected and has $n - 1$ edges. Therefore it is a tree.  \hfill \Box
Maximum weight trees

\( G = (V, E) \) is a connected graph.

\( w : E \rightarrow \mathbb{R} \). \( w(e) \) is the weight of edge \( e \).

For spanning tree \( T \), \( w(T) = \sum_{e \in T} w(e) \).

**Problem:** find a spanning tree of maximum weight.
Greedy Algorithm

Sort edges so that $E = \{e_1, e_2, \ldots, e_m\}$ where

$$w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m).$$

begin

$T := \emptyset$

for $i = 1, 2, \ldots, m$ do

begin

if $T + e_i$ does not contain a cycle

then $T \leftarrow T + e_i$

end

Output $T$

end

Greedy always adds the maximum weight edge which does not make a cycle with previously chosen edges.
**Theorem 4** The tree constructed by GREEDY has maximum weight.

**Proof** Let the edges of the greedy tree be 
\(e_1^*, e_2^*, \ldots, e_{n-1}^*\), in order of choice. Note that \(w(e_i^*) \geq w(e_{i+1}^*)\) since neither makes a cycle with \(e_1^*, e_2^*, \ldots, e_{i-1}^*\).

Let \(f_1, f_2, \ldots, f_{n-1}\) be the edges of any other tree where \(w(f_1) \geq w(f_2) \geq \cdots w(f_{n-1})\).

We show that

\[
w(e_i^*) \geq w(f_i) \quad 1 \leq i \leq n - 1. \tag{1}
\]
Suppose (1) is false. There exists $k > 0$ such that

$$w(e_i^*) \geq w(f_i), \ 1 \leq i < k \text{ and } w(e_k^*) < w(f_k).$$

Each of $f_i, 1 \leq i \leq k$ is either a member of
\{e_1^*, e_2^*, \ldots , e_{k-1}^*\} or makes a cycle with
\{e_1^*, e_2^*, \ldots , e_{k-1}^*\}. Otherwise one of them would have been chosen in preference to $e_k^*$.

Let the components of the forest
$(V, \{e_1^*, e_2^*, \ldots , e_{k-1}^*\})$ be $C_1, C_2, \ldots , C_{n-k+1}$. Each $f_i, 1 \leq i \leq k$ has both of its endpoints in the same component.
Let \( \mu_i \) be the number of \( f_j \) which have both endpoints in \( C_i \) and let \( \nu_i \) be the number of vertices of \( C_i \). Then

\[
\mu_1 + \mu_2 + \cdots + \mu_{n-k+1} = k \tag{2}
\]
\[
\nu_1 + \nu_2 + \cdots + \nu_{n-k+1} = n \tag{3}
\]

It follows from (2) and (3) that there exists \( t \) such that

\[
\mu_t \geq \nu_t. \tag{4}
\]

[Otherwise

\[
\sum_{i=1}^{n-k+1} \mu_i \leq \sum_{i=1}^{n-k+1} (\nu_i - 1)
\]

\[
= \sum_{i=1}^{n-k+1} \nu_i - (n-k+1)
\]

\[
= k - 1.
\]

] But (4) implies that the edges \( f_j \) such that \( f_j \subseteq C_t \) contain a cycle. \( \square \)
How many trees? – Cayley’s Formula

n=4

n=5

n=6
Prüfer’s Correspondence

There is a 1-1 correspondence \( \phi_V \) between spanning trees of \( K_N \) (the complete graph with vertex set \( V \)) and sequences \( V^{n-2} \). Thus for \( n \geq 2 \)

\[
\tau(K_n) = n^{n-2} \quad \text{Cayley’s Formula.}
\]

Assume some arbitrary ordering \( V = \{v_1 < v_2 < \cdots < v_n\} \).

\( \phi_V(T) \):
begin
\[ T_1 := T; \]
for \( i = 1 \) to \( n - 2 \) do
begin
\[ s_i := \text{neighbour of least leaf } \ell_i \text{ of } T_i. \]
\[ T_{i+1} = T_i - \ell_i. \]
end
\( \phi_V(T) = s_1 s_2 \cdots s_{n-2} \)
end
6,4,5,14,2,6,11,14,8,5,11,4,2
Lemma 3 \( v \in V(T) \) appears exactly \( d_T(v) - 1 \) times in \( \phi_V(T) \).

Proof. Assume \( n = |V(T)| \geq 2 \). By induction on \( n \).

\( n = 2: \phi_V(T) = \Lambda = \) empty string.

Assume \( n \geq 3: \)

\[ \phi_V(T) = s_1 \phi_{V_1}(T_1) \text{ where } V_1 = V - \{s_1\}. \]

\( s_1 \) appears \( d_{T_1}(s_1) - 1 + 1 = d_T(s_1) - 1 \) times – induction.

\( v \neq s_1 \) appears \( d_{T_1}(v) - 1 = d_T(v) - 1 \) times – induction.

\( \square \)
Construction of $\phi_V^{-1}$

Inductively assume that for all $|X| < n$ there is an inverse function $\phi_X^{-1}$. (True for $n = 2$).
Now define $\phi_V^{-1}$ by

$$\phi_V^{-1}(s_1s_2\ldots s_{n-2}) = \phi_{V_1}^{-1}(s_2\ldots s_{n-2}) \text{ plus edge } s_1\ell_1,$$

where $\ell_1 = \min\{s : s \notin s_1, s_2, \ldots s_{n-2}\}$ and $V_1 = V - \{\ell_1\}$. Then

$$\phi_V(\phi_V^{-1}(s_1s_2\ldots s_{n-2})) = s_1\phi_{V_1}(\phi_{V_1}^{-1}(s_2\ldots s_{n-2}))$$

$$= s_1s_2\ldots s_{n-2}.$$

Thus $\phi_V$ has an inverse and the correspondence is established.
Number of trees with a given degree sequence

**Corollary 5** If \( d_1 + d_2 + \cdots + d_n = 2n - 2 \) then the number of spanning trees of \( K_n \) with degree sequence \( d_1, d_2, \ldots, d_n \) is

\[
\binom{n - 2}{d_1 - 1, d_2 - 1, \ldots, d_n - 1} = \frac{(n - 2)!}{(d_1 - 1)! (d_2 - 1)! \cdots (d_n - 1)!}.
\]

**Proof** From Prüfer’s correspondence and Lemma 3 this is the number of sequences of length \( n - 2 \) in which 1 appears \( d_1 - 1 \) times, 2 appears \( d_2 - 1 \) times and so on.

\( \Box \)