PIGEON HOLE PRINCIPLE
If $f : [m] \rightarrow [n]$ then there exists $i \in [n]$ such that

$$|f^{-1}(i)| \geq \lceil m/n \rceil.$$ 

Informally: If $m$ pigeons are to be placed in $n$ pigeon-holes, at least one hole will end up with at least $\lceil m/n \rceil$ pigeons.
Example 1:

There are at least $10^4$ people in China who have exactly the same number of strands of hair.

**Proof.** A human may have $0 \leq x \leq 10^5 - 1$ hair strands. There are more then $10^9$ people living in China. Label a ‘hole’ by the number of hair strands and put a person in hole $i$ if she/he has exactly $i$ hair strands. There must be at least one hole with $10^4$ people or more people. □
Positive integers $n$ and $k$ are co-prime if their largest common divisor is 1.

**Example 2.** If we take an arbitrary subset $A$ of $n + 1$ integers from the set $[2n] = \{1, \ldots, 2n\}$ it will contain a pair of co-prime integers.

If we take the $n$ even integers between 1 and $2n$. This set of $n$ elements does not contain a pair of mutually prime integers. Thus we cannot replace the $n + 1$ by $n$ in the statement. We say that the statement is *tight*. 

Pigeon Hole Principle
Define the holes as sets \( \{1, 2\}, \{3, 4\}, \ldots \{2n - 1, 2n\} \). Thus \( n \) holes are defined.

If we place the \( n + 1 \) integers of \( A \) into their corresponding holes – by the pigeon-hole principle – there will be a hole, which will contain two numbers.

This means, that \( A \) has to contain two consecutive integers, say, \( x \) and \( x + 1 \). But two such numbers are always co-prime.

If some integer \( y \neq 1 \) divides \( x \), i.e., \( x = ky \), then \( x + 1 = ky + 1 \) and this is not divisible by \( y \). □
We have two disks, each partitioned into 200 sectors of the same size. 100 of the sectors of Disk 1 are coloured Red and 100 are colored Blue. The 200 sectors of Disk 2 are arbitrarily coloured Red and Blue.

It is always possible to place Disk 2 on top of Disk 1 so that the centres coincide, the sectors line up and at least 100 sectors of Disk 2 have the same colour as the sector underneath them.

Fix the position of Disk 1. There are 200 positions for Disk 2 and let \( q_i \) denote the number of matches if Disk 2 is placed in position \( i \). Now for each sector of Disk 2 there are 100 positions \( i \) in which the colour of the sector underneath it coincides with its own.
Therefore

\[ q_1 + q_2 + \cdots + q_{200} = 200 \times 100 \]  

(1)

and so there is an \( i \) such that \( q_i \geq 100 \).

Explanation of (1).
Consider 0-1 \( 200 \times 200 \) matrix \( A(i, j) \) where \( A(i, j) = 1 \) iff sector \( j \) lies on top of a sector with the same colour when in position \( i \). Row \( i \) of \( A \) has \( q_i \) 1’s and column \( j \) of \( A \) has 100 1’s. The LHS of (1) counts the number of 1’s by adding rows and the RHS counts the number of 1’s by adding columns.

Pigeon Hole Principle
Alternative solution: Place Disk 2 randomly on Disk 1 so that the sectors align. For $i = 1, 2, \ldots, 200$ let

$$X_i = \begin{cases} 
1 & \text{sector } i \text{ of disk 2 is on sector of disk 1 of same color} \\
0 & \text{otherwise}
\end{cases}$$

We have

$$E(X_i) = \frac{1}{2} \quad \text{for } i = 1, 2, \ldots, 200.$$ 

So if $X = X_1 + \cdots + X_{200}$ is the number of sectors sitting above sectors of the same color, then $E(X) = 100$ and there must exist at least one way to achieve 100.
Theorem

(Erdős-Szekeres) An arbitrary sequence of integers \((a_1, a_2, \ldots, a_{k^2+1})\) contains a monotone subsequence of length \(k + 1\).

Proof. Let \((a_i, a_i^1, a_i^2, \ldots, a_i^{\ell-1})\) be the longest monotone increasing subsequence of \((a_1, \ldots, a_{k^2+1})\) that starts with \(a_i, (1 \leq i \leq k^2 + 1)\), and let \(\ell(a_i)\) be its length.

If for some \(1 \leq i \leq k^2 + 1, \ell(a_i) \geq k + 1\), then \((a_i, a_i^1, a_i^2, \ldots, a_i^{\ell-1})\) is a monotone increasing subsequence of length \(\geq k + 1\).

So assume that \(\ell(a_i) \leq k\) holds for every \(1 \leq i \leq k^2 + 1\).
Consider $k$ holes $1, 2, \ldots, k$ and place $i$ into hole $\ell(a_i)$.

There are $k^2 + 1$ subsequences and $\leq k$ non-empty holes (different lengths), so by the pigeon-hole principle there will exist $\ell^*$ such that there are (at least) $k + 1$ indices $i_1 < i_2 < \cdots < i_{k+1}$ such that $\ell(a_{i_t}) = \ell^*$ for $1 \leq t \leq k + 1$.

Then we must have $a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_{k+1}}$.

Indeed, assume to the contrary that $a_{i_m} < a_{i_n}$ for some $1 \leq m < n \leq k + 1$. Then $a_{i_m} \leq a_{i_n} \leq a_{i_n}^1 \leq a_{i_n}^2 \leq \cdots \leq a_{i_n}^{\ell^* - 1}$, i.e., $\ell(a_{i_m}) \geq \ell^* + 1$, a contradiction. \qed
The sequence

\[ n, n-1, \ldots, 1, 2n, 2n-1, \ldots, n+1, \ldots, n^2, n^2-1, \ldots, n^2-n+1 \]

has no monotone subsequence of length \( n+1 \) and so the Erdős-Szekeres result is best possible.
Let $P_1, P_2, \ldots, P_n$ be $n$ points in the unit square $[0, 1]^2$. We will show that there exist $i, j, k \in [n]$ such that the triangle $P_i P_j P_k$ has area

$$\leq \frac{1}{2(\lfloor \sqrt{(n-1)/2} \rfloor)^2} \sim \frac{1}{n}$$

for large $n$. 

Pigeon Hole Principle
Let $m = \lfloor \sqrt{(n-1)/2} \rfloor$ and divide the square up into $m^2 < \frac{n}{2}$ subsquares. By the pigeonhole principle, there must be a square containing $\geq 3$ points. Let 3 of these points be $P_iP_jP_k$. The area of the corresponding triangle is at most one half of the area of an individual square.