GAME THEORY

Class notes for Math 167, Fall 2000

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Part I. Impartial Combinatorial Games

1. Take-Away Games.

Combinatorial games are two-person games with perfect information and no chance moves, and with a win-or-lose outcome. Such a game is determined by a set of positions, including an initial position, and the player whose turn it is to move. Play moves from one position to another, with the players usually alternating moves, until a terminal position is reached. A terminal position is one from which no moves are possible. Then one of the players is declared the winner and the other the loser.

There are two main references for the material on combinatorial games. One is the research book, *On Numbers and Games* by J. H. Conway, Academic Press, 1976. This book introduced many of the basic ideas of the subject and led to a rapid growth of the area that continues today. The other reference, more appropriate for this class, is the two-volume book, *Winning Ways for your mathematical plays* by Berlekamp, Conway and Guy, Academic Press, 1982, in paperback. There are many interesting games described in this book and much of it is accessible to the undergraduate mathematics student. This theory may be divided into two parts, **impartial games** in which the set of moves available from any given position is the same for both players, and **partizan games** in which each player has a different set of possible moves from a given position. Games like chess or checkers in which one player moves the white pieces and the other moves the black pieces are partizan. In Part I, we treat only the theory of impartial games. An elementary introduction to impartial combinatorial games is given in the book *Fair Game* by Richard K. Guy, published in the COMAP Mathematical Exploration Series, 1989. We start with a simple example.

1.1 A Simple Take-Away Game. Here are the rules of a very simple impartial combinatorial game of removing chips from a pile of chips.

(1) *There are two players. We label them I and II.*

(2) *There is a pile of 21 chips in the center of a table.*

(3) *A move consists of removing one, two, or three chips from the pile. At least one chip must be removed, but no more than three may be removed.*

(4) *Players alternate moves with Player I starting.*

(5) *The player that removes the last chip wins. (The last player to move wins. If you can't move, you lose.)*

How can we analyze this game? Can one of the players force a win in this game? Which player would you rather be, the player who starts or the player who goes second? What is a good strategy?

We analyze this game from the end back to the beginning. This method is sometimes called **backward induction**.

If there are just one, two, or three chips left, the player who moves next wins simply by taking all the chips.

Suppose there are four chips left. Then the player who moves next must leave either one, two or three chips in the pile and his opponent will be able to win. So four chips left is a loss for the next player to move and a win for the previous player, i.e. the one who just moved.

With 5, 6, or 7 chips left, the player who moves next can win by moving to the position with four chips left.

With 8 chips left, the next player to move must leave 5, 6, or 7 chips, and so the previous player can win.

We see that positions with $0, 4, 8, 12, 16, \ldots$ chips are target positions; we would like to move into them. We may now analyze the game with 21 chips. Since 21 is not divisible by 4, the first player to move can win. The unique optimal move is to take one chip and leave 20 chips which is a target position.

1.2 What is a Combinatorial Game? We now define the notion of a combinatorial game more precisely. It is a game that satisfies the following conditions.

(1) *There are two players.*

(2) *There is a set, usually finite, of possible positions of the game.*

(3) *The rules of the game specify for both players and each position which moves to other positions are legal moves. If the rules make no distinction between the players, that is if both players have the same options of moving from each position, the game is called* **impartial***; otherwise, the game is called* **partizan**.

(4) *The players alternate moving.*

(5) *The game ends when a position is reached from which no moves are possible for the player whose turn it is to move. Under the* **normal play rule***, the last player to move wins. Under the* **misère play rule** *the last player to move loses.*

If the game never ends, it is declared a draw. However, we shall nearly always add the following condition, called **the ending condition**. This eliminates the possibility of a draw.

(6) *The game ends in a finite number of moves no matter how it is played.*

It is important to note what is omitted in this definition. No random moves such as the rolling of dice or the dealing of cards are allowed. This rules out games like backgammon and poker. A combinatorial game is a game of perfect information: simultaneous moves and hidden moves are not allowed. This rules out battleship and scissors-paper-rock. No draws in a finite number of moves are possible. This rules out tic-tac-toe. In these notes, we restrict attention to impartial games, generally under the normal play rule.

1.3 P-positions, N-positions. Returning to the take-away game of Section 1.1, we see that $0, 4, 8, 12, 16, \ldots$ are positions that are winning for the **P**revious player (the player who just moved) and that $1, 2, 3, 5, 6, 7, 9, 10, 11, \ldots$ are winning for the **N**ext player to move. The former are called P-positions, and the latter are called N-positions. The

P-positions are just those with a number of chips divisible by 4, called target positions in Section 1.1.

In impartial combinatorial games, one can find in principle which positions are P-positions and which are N-positions by (possibly transfinite) induction using the following labeling procedure starting at the terminal positions. We say a position in a game is a **terminal position**, if no moves from it are possible. This algorithm is just the method we used to solve the take-away game of Section 1.1.

Step 1: Label every terminal position as a P-position.

Step 2: Label every position that can reach a labelled P-position in one move as an N-position.

Step 3: Find those positions whose only moves are to labelled N-positions; label such positions as P-positions.

Step 4: If no new P-positions were found in step 3, stop; otherwise return to step 2.

It is easy to see that the strategy of moving to P-positions wins. From a P-position, your opponent can move only to an N-position (3). Then you may move back to a P-position (2). Eventually the game ends at a terminal position and since this is a P-position, you win (1).

Here is a characterization of P-positions and N-positions that is valid for impartial combinatorial games satisfying the ending condition, under the normal play rule.

P-positions and N-positions are defined recursively by the following three statements.
(1) *All terminal positions are P-positions.*
(2) *From every N-position, there is at least one move to a P-position.*
(3) *From every P-position, every move is to an N-position.*

For games using the misére play rule, condition (1) should be replaced by the condition that all terminal positions are N-positions.

1.4 Subtraction Games. Let us now consider a class of combinatorial games that contains the take-away game of Section 1.1 as a special case. Let S be a set of positive integers. The subtraction game with subtraction set S is played as follows. From a pile with a large number, say n, of chips, two players alternate moves. A move consists of removing s chips from the pile where $s \in S$. Last player to move wins.

The take-away game of Section 1.1 is the subtraction game with subtraction set $S = \{1, 2, 3\}$. In Exercise 1.2, you are asked to analyze the subtraction game with subtraction set $S = \{1, 2, 3, 4, 5, 6\}$.

For illustration, let us analyze the subtraction game with subtraction set $S = \{1, 3, 4\}$ by finding its P-positions. There is exactly one terminal position, namely 0. Then 1, 3, and 4 are N-positions, since they can be moved to 0. But 2 then must be a P-position since the only legal move from 2 is to 1, which is an N-position. Then 5 and 6 must be N-positions since they can be moved to 2. Now we see that 7 must be a P-position since the only moves from 7 are to 6, 4, or 3, all of which are N-positions.

Now we continue similarly: we see that 8, 10 and 11 are N-positions, 9 is a P-position, 12 and 13 are N-positions and 14 is a P-position. This extends by induction. We find that the set of P-positions is $P = \{0, 2, 7, 9, 14, 16, \ldots\}$, the set of nonnegative integers leaving remainder 0 or 2 when divided by 7. The set of N-positions is the complement, $N = \{1, 3, 4, 5, 6, 8, 10, 11, 12, 13, 15, \ldots\}$.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14...
position	P	N	P	N	N	N	N	P	N	P	N	N	N	N	P...

The pattern $PNPNNNN$ of length 7 repeats forever.

Who wins the game with 100 chips, the first player or the second? The P-positions are the numbers equal to 0 or 2 modulus 7. Since 100 has remainder 2 when divided by 7, 100 is a P-position; the second player to move can win with optimal play.

1.5 Exercises.

1. Consider the misère version of the take-away game of Section 1.1, where the last player to move loses. The object is to force your opponent to take the last chip. Analyze this game. What are the target positions (P-positions)? (You can play the normal version of the game at http://207.106.82.89/puzzles/23match/23match.htm .)

2. Generalize the Take-Away Game: (a) Suppose in a game with a pile containing a large number of chips, you can remove any number from 1 to 6 chips at each turn. What is the winning strategy? What are the P-positions? (b) If there are initially 31 chips in the pile, what is your winning move, if any?

3. **The Thirty-one Game.** (Geoffrey Mott-Smith (1954)) From a deck of cards, take the Ace, 2, 3, 4, 5, and 6 of each suit. These 24 cards are laid out face up on a table. The players alternate turning over cards and the sum of the turned over cards is computed as play progresses. Each Ace counts as one. The player who first makes the sum go above 31 loses. It would seem that this is equivalent to the game of the previous exercise played on a pile of 31 chips. But there is a catch. No integer may be chosen more than four times.
(a) If you are the first to move, and if you use the strategy found in the previous exercise, what happens if the opponent keeps choosing 4?
(b) Nevertheless, the first player can win with optimal play. How?

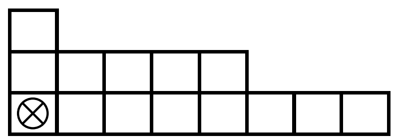
4. Find the set of P-positions for the subtraction games with subtraction sets
(a) $S = \{1, 3, 5, 7\}$.
(b) $S = \{1, 3, 6\}$.
(c) $S = \{1, 2, 4, 8, 16, \ldots\}$ = all powers of 2.
(d) Who wins each of these games if play starts at 100 chips, the first player or the second?

5. **Empty and Divide.** There are two boxes. Initially, one box contains m chips and the other contains n chips. Such a position is denoted by (m, n), where $m > 0$ and $n > 0$. The two players alternate moving. A move consists of emptying one of the boxes, and dividing the contents of the other between the two boxes with at least one chip in each box. There is a unique terminal position, namely $(1, 1)$. Last player to move wins. Find all P-positions.

6. **Chomp!** A game invented by Fred. Schuh (1952) in an arithmetical form was discovered independently in a completely different form by David Gale (1974). Gale's version of the game involves removing squares from a rectangular board, say an m by n

board. A move consists in taking a square and removing it and all squares to the right and above. Players alternate moves, and the person to take square $(1,1)$ loses. The name "Chomp" comes from imagining the board as a chocolate bar, and moves involving breaking off some corner squares to eat. The square $(1,1)$ is poisoned though; the player who chomps it loses. You can play this game on the web at
http://207.106.82.89/puzzles/chomp/chomp.htm .

For example, starting with an 8 by 3 board, suppose the first player chomps at $(6,2)$ gobbling 6 pieces, and then second player chomps at $(2,3)$ gobbling 4 pieces, leaving the following board, where $\otimes$ denotes the poisoned piece.



(a) Show that this position is a N-position, by finding a winning move for the first player. (It is unique.)

(b) It is known that the first player can win all *rectangular* starting positions. The proof, though ingenious, is not hard. However, it is an "existence" proof. It shows that there is a winning strategy for the first player, but gives no hint on how to find the first move! See if you can find the proof. Here is a hint: Does removing the upper right corner constitute a winning move?

7. **Dynamic subtraction.** One can enlarge the class of subtraction games by letting the subtraction set depend on the last move of the opponent. Many early examples appear in Chapter 12 of Schuh (1968). Here are two other examples. (For a generalization, see Schwenk (1970).)
(a) There is one pile of n chips. The first player to move may remove as many chips as desired, at least one chip but not the whole pile. Thereafter, the players alternate moving, each player not being allowed to remove more chips than his opponent took on the previous move. What is an optimal move for the first player if $n = 44$? For what values of n does the second player have a win?
(b) **Fibonacci Nim.** (Whinihan (1963)) The same rules as in (a), except that a player may take at most twice the number of chips his opponent took on the previous move. The analysis of this game is more difficult than the game of part (a) and depends on the sequence of numbers named after Leonardo Pisano Fibonacci, which may be defined as $F_1 = 1$, $F_2 = 2$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. The Fibonacci sequence is thus: $1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$. The solution is facilitated by

Zeckendorf's Theorem. *Every positive integer can be written uniquely as a sum of distinct non-neighboring Fibonacci numbers.*

There may be many ways of writing a number as a sum of Fibonacci numbers, but there is only one way of writing it as a sum of non-neighboring Fibonacci numbers. Thus, 43=34+8+1 is the unique way of writing 43, since although 43=34+5+3+1, 5 and 3 are neighbors. What is an optimal move for the first player if $n = 43$? For what values of n does the second player have a win?

8. **The *SOS* Game.** (From the 28th Annual USA Mathematical Olympiad, 1999) The board consists of a row of n squares, initially empty. Players take turns selecting an empty square and writing either an S or an O in it. The player who first succeeds in completing SOS in consecutive squares wins the game. If the whole board gets filled up without an SOS appearing consecutively anywhere, the game is a draw.
(a) Suppose $n = 4$ and the first player puts an S in the first square. Show the second player can win.
(b) Show that if $n = 7$, the first player can win the game.
(c) Show that if $n = 2000$, the second player can win the game.
(d) Who wins the game if $n = 14$?

2. The Game of Nim.

The most famous take-away game is the game of Nim, played as follows. There are three piles of chips containing x_1, x_2, and x_3 chips respectively. (Piles of sizes 5, 7, and 9 make a good game.) Two players take turns moving. Each move consists of selecting one of the piles and removing chips from it. You may not remove chips from more than one pile in one turn, but from the pile you selected you may remove as many chips as desired, from one chip to the whole pile. The winner is the player who removes the last chip. You can play this game on the web with four piles at Game of Nim (http://www.thebestweb.com/iss/testOfGame.asp), or with even more piles at Nim Game (http://www.dotsphinx.com/nim/).

2.1 Preliminary Analysis. There is exactly one terminal position, namely $(0,0,0)$, which is therefore a P-position. The solution to one-pile Nim is trivial: you simply remove the whole pile. Any position with exactly one non-empty pile, say $(0,0,x)$ with $x > 0$ is therefore an N-position. Consider two-pile Nim. It is easy to see that the P-positions are those for which the two piles have an equal number of chips, $(0,1,1)$, $(0,2,2)$, etc. This is because if it is the opponent's turn to move from such a position, he must change to a position in which the two piles have an unequal number of chips, and then you can immediately return to a position with an equal number of chips (perhaps the terminal position).

If all three piles are non-empty, the situation is more complicated. Clearly, $(1,1,1)$, $(1,1,2)$, $(1,1,3)$ and $(1,2,2)$ are all N-positions because they can be moved to $(1,1,0)$ or $(0,2,2)$. The next simplest position is $(1,2,3)$ and it must be a P-position because it can only be moved to one of the previously discovered N-positions. We may go on and discover that the next most simple P-positions are $(1,4,5)$, and $(2,4,6)$, but it is difficult to see how to generalize this. Is $(5,7,9)$ a P-position? Is $(15,23,30)$ a P-position?

If you go on with the above analysis, you may discover a pattern. But to save us some time, I will describe the solution to you. Since the solution is somewhat fanciful and involves something called nim-sum, the validity of the solution is not obvious. Later, we prove it to be valid using the elementary notions of P-position and N-position.

2.2 Nim-Sum. The nim-sum of two non-negative integers is their addition without carry in base 2. Let us make this notion precise.

Every non-negative integer x has a unique base 2 representation of the form $x = x_m 2^m + x_{m-1} 2^{m-1} + \cdots + x_1 2 + x_0$ for some m, where each x_i is either zero or one. We use the notation $(x_m x_{m-1} \cdots x_1 x_0)_2$ to denote this representation of x to the base two. Thus, $22 = 1 \cdot 16 + 0 \cdot 8 + 1 \cdot 4 + 1 \cdot 2 + 0 \cdot 1 = (10110)_2$. The nim-sum of two integers is found by expressing the integers to base two and using addition modulo 2 on the corresponding individual components:

Definition. *The nim-sum of $(x_m \cdots x_0)_2$ and $(y_m \cdots y_0)_2$ is $(z_m \cdots z_0)_2$, and we write $(x_m \cdots x_0)_2 \oplus (y_m \cdots y_0)_2 = (z_m \cdots z_0)_2$, where for all k, $z_k = x_k + y_k$ (mod 2), that is, $z_k = 1$ if $x_k + y_k = 1$ and $z_k = 0$ otherwise.*

For example, $(10110)_2 \oplus (110011)_2 = (100101)_2$. This says that $22 \oplus 51 = 37$. This is easier to see if the numbers are written vertically (we also omit the parentheses for clarity):

$$\begin{array}{rr} 22 = & 10110_2 \\ 51 = & 110011_2 \\ \hline \text{nim-sum} = & 100101_2 = 37 \end{array}$$

Nim-sum is associative (i.e. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$) and commutative (i.e. $x \oplus y = y \oplus x$), since addition modulo 2 is. Thus we may write $x \oplus y \oplus z$ without specifying the order of addition. Furthermore, 0 is an identity for addition ($0 \oplus x = x$), and every number is its own inverse ($x \oplus x = 0$), so that the cancellation law holds: $x \oplus y = x \oplus z$ implies $y = z$. (If $x \oplus y = x \oplus z$, then $x \oplus x \oplus y = x \oplus x \oplus z$, and so $y = z$.)

Thus, nim-sum has a lot in common with ordinary addition, but what does it have to do with playing the game of Nim? The answer is contained in the following theorem of C. L. Bouton (1902).

Theorem 1. *A position, (x_1, x_2, x_3), in Nim is a P-position if and only if the nim-sum of its components is zero, $x_1 \oplus x_2 \oplus x_3 = 0$.*

As an example, take the position $(x_1, x_2, x_3) = (13, 12, 8)$. Is this a P-position? If not, what is a winning move? We compute the nim-sum of 13, 12 and 8:

$$\begin{array}{rr} 13 = & 1101_2 \\ 12 = & 1100_2 \\ 8 = & 1000_2 \\ \hline \text{nim-sum} = & 1001_2 = 9 \end{array}$$

Since the nim-sum is not zero, this is an N-position according to Theorem 1. Can you find a winning move? You must find a move to a P-position, that is, to a position with an even number of 1's in each column. One such move is to take away 9 chips from the pile of 13, leaving 4 there. The resulting position has nim-sum zero:

$$\begin{array}{rr} 4 = & 100_2 \\ 12 = & 1100_2 \\ 8 = & 1000_2 \\ \hline \text{nim-sum} = & 0000_2 = 0 \end{array}$$

Another winning move is to subtract 7 chips from the pile of 12, leaving 5. Check it out. There is also a third winning move. Can you find it?

2.3 Nim with a Larger Number of Piles. We saw that 1-pile nim is trivial, and that 2-pile nim is easy. Since 3-pile nim is much more complex, we might expect 4-pile nim to be much harder still. But that is not the case. Theorem 1 also holds for a larger number of piles! A nim position with four piles, (x_1, x_2, x_3, x_4), is a P-position if and only if $x_1 \oplus x_2 \oplus x_3 \oplus x_4 = 0$. The proof below works for an arbitrary finite number of piles.

2.4 Proof of Bouton's Theorem. Let $\mathcal{P}$ denote the set of Nim positions with nim-sum zero, and let $\mathcal{N}$ denote the complement set, the set of positions of positive nim-sum. We check the three conditions of the definition in Section 1.3.

(1) *All terminal positions are in $\mathcal{P}$.* That's easy. The only terminal position is the position with no chips in any pile, and $0 \oplus 0 \oplus \cdots = 0$.

(2) *From each position in $\mathcal{N}$, there is a move to a position in $\mathcal{P}$.* Here's how we construct such a move. Form the nim-sum as a column addition, and look at the leftmost (most significant) column with an odd number of 1's. Change any of the numbers that have a 1 in that column to a number such that there are an even number of 1's in each column. This makes that number smaller because the 1 in the most significant position changes to a 0. Thus this is a legal move to a position in $\mathcal{P}$.

(3) *Every move from a position in $\mathcal{P}$ is to a position in $\mathcal{N}$.* If $(x_1, x_2, \ldots)$ is in $\mathcal{P}$ and x_1 is changed to $x_1' < x_1$, then we cannot have $x_1 \oplus x_2 \oplus \cdots = 0 = x_1' \oplus x_2 \oplus \cdots$, because the cancellation law would imply that $x_1 = x_1'$. So $x_1' \oplus x_2 \oplus \cdots \neq 0$, implying that $(x_1', x_2, \ldots)$ is in $\mathcal{N}$.

These three properties show that $\mathcal{P}$ is the set of P-positions. ∎

It is interesting to note from (2) that in the game of nim the number of winning moves from an N-position is equal to the number of 1's in the leftmost column with an odd number of 1's. In particular, there is always an odd number of winning moves.

2.5 Misère Nim. What happens when we play nim under the misère play rule? Can we still find who wins from an arbitrary position, and give a simple winning strategy? This is one of those questions that at first looks hard, but after a little thought turns out to be easy.

Here is Bouton's method for playing misère nim optimally. Play it as you would play nim under the normal play rule as long as there are at least two heaps of size greater than one. When your opponent finally moves so that there is exactly one pile of size greater than one, reduce that pile to zero or one, whichever leaves an *odd* number of piles of size one remaining.

This works because your optimal play in nim never requires you to leave exactly one pile of size greater than one (the nim sum must be zero), and your opponent cannot move from two piles of size greater than one to no piles greater than one. So eventually the game drops into a position with exactly one pile greater than one and it must be your turn to move.

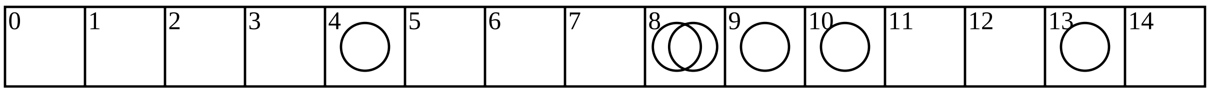
A similar analysis works in many other games. But in general the misère play theory is much more difficult than the normal play theory. Some games have a fairly simple normal play theory but an extraordinarily difficult misère theory, such as the games of Kayles and Dawson's chess, presented in Section 1.4.

2.6 Exercises.

1. (a) What is the nim-sum of 27 and 17?
(b) The nim-sum of 38 and x is 25. Find x.

2. Find all winning moves in the game of nim,
(a) with three piles of 12, 19, and 27 chips.
(b) with four piles of 13, 17, 19, and 23 chips.
(c) What is the answer to (a) and (b) if the misére version of nim is being played?

3. **Nimble.** Nimble is played on a game board consisting of a line of squares labelled: 0, 1, 2, 3, ... A finite number of coins is placed on the squares with possibly more than one coin on a single square. A move consists in taking one of the coins and moving it to any square to the left, possibly moving over some of the coins, and possibly onto a square already containing one or more coins. The players alternate moves and the game ends when all coins are on the square labelled 0. The last player to move wins. Show that this game is just nim in disguise. In the following position with 6 coins, who wins, the next player or the previous player? If the next player wins, find a winning move.



4. **Turning Turtles.** Another class of games, due to H. W. Lenstra, is played with a long line of coins, with moves involving turning over some coins from heads to tails or from tails to heads. See *Winning Ways*, Chapter 14 for some of the remarkable theory. Here is a simple example called Turning Turtles.

A horizontal line of n coins is laid out randomly with some coins showing heads and some tails. A move consists of turning over one of the coins from heads to tails, and in addition, if desired, turning over one other coin to the left of it (from heads to tails or tails to heads). For example consider the sequence of $n = 13$ coins:

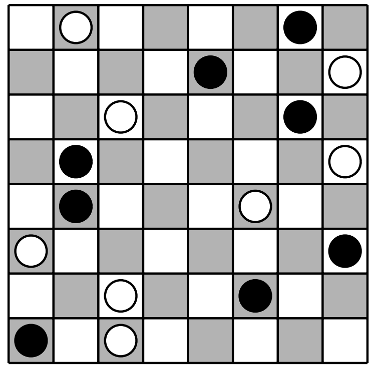
T	H	T	T	H	T	T	T	H	H	T	H	T
1	2	3	4	5	6	7	8	9	10	11	12	13

One possible move in this position is to turn the coin in place 9 from heads to tails, and also the coin in place 4 from tails to heads.
(a) Show that this game is just nim in disguise if an H in place n is taken to represent a nim pile of n chips.
(b) Assuming (a) to be true, find a winning move in the above position.
(c) Try this game and some other related games at
http://www.chlond.demon.co.uk/Coins.html .

5. **Northcott's Game.** A position in Northcott's game is a checkerboard with one black and one white checker on each row. "White" moves the white checkers and "Black" moves the black checkers. A checker may move any number of squares along its row, but may not jump over or onto the other checker. Players move alternately and the last to move wins. Try out this game at http://www.chlond.demon.co.uk/Northcott.html .

Note two points:
1. This is a partizan game, because Black and White have different moves from a given position.
2. This game does not satisfy the ending condition, (6) of Section 1.2. The players could move around endlessly.

Nevertheless, knowing how to play nim is a great advantage in this game. In the position below, who wins, Black or White? or does it depend on who moves first?



6. **Staircase Nim.** (Sprague (1937)) A staircase of n steps contains coins on some of the steps. Let $(x_1, x_2, \ldots, x_n)$ denote the position with x_j coins on step j, $j = 1, \ldots, n$. A move of staircase nim consists of moving any positive number of coins from any step, j, to the next lower step, $j - 1$. Coins reaching the ground (step 0) are removed from play. Such a move would take, say, x chips from step j, where $1 \le x \le x_j$, and put them on step $j - 1$, leaving $x_j - x$ coins on step j and resulting in $x_{j-1} + x$ coins on step $j - 1$. The game ends when all coins are on the ground. Players alternate moves and the last to move wins.

Show that $(x_1, x_2, \ldots, x_n)$ is a P-position if and only if the numbers of coins on the odd numbered steps, $(x_1, x_3, \ldots, x_k)$ where $k = n$ if n is odd and $k = n - 1$ if n is even, forms a P-position in ordinary nim.

7. **Moore's Nim$_k$.** A generalization of nim with a similar elegant theory was proposed by E. H. Moore (1910), called Nim$_k$. There are n piles of chips and play proceeds as in nim except that in each move a player may remove as many chips as desired from any k piles, where k is fixed. At least one chip must be taken from some pile. For $k = 1$ this reduces to ordinary nim, so ordinary nim is Nim$_1$.

Moore's Theorem states that a position $(x_1, x_2, \ldots, x_n)$, is a P-position in Nim$_k$ if and only if when x_1 to x_n are expanded in base 2 and added in base $k + 1$ without carry, the result is zero. (In other words, the number of 1's in each column must be divisible by $k + 1$.)

(a) Consider the game of Nimble of Exercise 3 but suppose that at each turn a player may move one or two coins to the left as many spaces as desired. Note that this is really Moore's Nim$_k$ with $k = 2$. Using Moore's Theorem, show that the Nimble position of Exercise 3 is an N-position, and find a move to a P-position.

(b) Prove Moore's Theorem.

3. Graph Games.

We now give an equivalent description of a combinatorial game as a game played on a directed graph. This will contain the games described in Sections 1 and 2. This is done by identifying positions in the game with vertices of the graph and moves of the game with edges of the graph. Then we will define a function known as the Sprague-Grundy function that contains more information than just knowing whether a position is a P-position or an N-position.

3.1 Games Played on Directed Graphs. We first give the mathematical definition of a directed graph.

Definition. *A* **directed graph**, G, *is a pair* (X, F) *where* X *is a nonempty set of vertices (***positions***) and* F *is a function that gives for each* $x \in X$ *a subset of* X, $F(x) \subset X$. *For a given* $x \in X$, $F(x)$ *represents the positions to which a player may move from* x *(called the* **followers** *of* x*). If* $F(x)$ *is empty,* x *is called a terminal position.*

A two-person win-lose game may be played on such a graph $G = (X, F)$ by stipulating a starting position $x_0 \in X$ and using the following rules:

(1) Player I moves first, starting at x_0.

(2) Players alternate moves.

(3) At position x, the player whose turn it is to move chooses a position $y \in F(x)$.

(4) The player who is confronted with a terminal position at his turn, and thus cannot move, loses.

As defined, graph games could continue for an infinite number of moves. To avoid this possibility and other problems, we restrict attention to graphs that have the property that no matter what starting point x_0 is used, there is a number n, possibly depending on x_0, such that every path from x_0 has length less than or equal to n. (A **path** is a sequence $x_0, x_1, x_2, \ldots, x_m$ such that $x_i \in F(x_{i-1})$ for all $i = 1, \ldots, m$, where m is the length of the path.) Such graphs are called **progressively bounded**. (If X itself is finite, this merely means that there are no **circuits**. A circuit is a path, $x_0, x_1, \ldots, x_m$, with $x_0 = x_m$ and distinct vertices $x_0, x_1, \ldots, x_{m-1}$, $m \geq 1$.)

As an example, the subtraction game with subtraction set $S = \{1, 2, 3\}$, analyzed in Section 1.1, that starts with a pile of n chips has a representation as a graph game. Here $X = \{0, 1, \ldots, n\}$ is the set of vertices. The empty pile is terminal, so $F(0) = \emptyset$, the empty set. We also have $F(1) = \{0\}$, $F(2) = \{0, 1\}$, and for $2 \leq k \leq n$, $F(k) = \{k-3, k-2, k-1\}$. This completely defines the game.

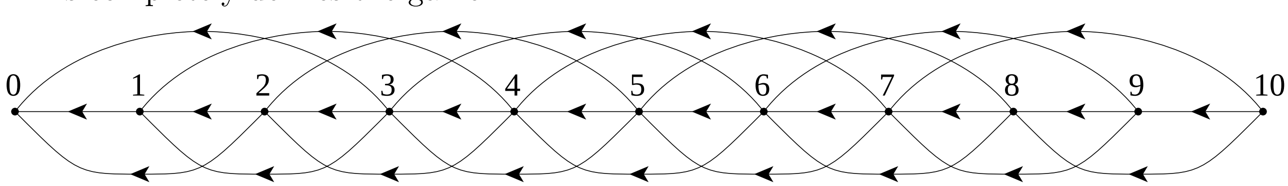


Fig. 3.1 The Subtraction Game with $S = \{1, 2, 3\}$.

It is useful to draw a representation of the graph. This is done using dots to represent vertices and lines to represent the possible moves. An arrow is placed on each line to indicate which direction the move goes. The graphic representation of this subtraction game played on a pile of 10 chips is given in Figure 3.1.

3.2 The Sprague-Grundy Function. Graph games may be analyzed by considering P-positions and N-positions. It may also be analyzed through the Sprague-Grundy function.

Definition. *The* **Sprague-Grundy function** *of a progressively bounded graph* (X, F) *is a function,* g*, defined on* X *taking non-negative integer values such that*

$$g(x) = \min\{n \geq 0 : n \neq g(y) \quad \textit{for} \quad y \in F(x)\}.$$

In words, $g(x)$ the smallest non-negative integer not found among the Sprague-Grundy values of the followers of x. If we define the **minimal excludant**, or **mex**, of a set of non-negative integers as the smallest non-negative integer not in the set, then we may write simply

$$g(x) = \text{mex}\{g(y) : y \in F(x)\}.$$

Note that $g(x)$ is defined recursively. That is, $g(x)$ is defined in terms of $g(y)$ for all followers y of x. Moreover, the recursion is self-starting. For terminal vertices, x, the definition implies that $g(x) = 0$, since $F(x) = \emptyset$ for terminal x. For non-terminal x, all of whose followers are terminal, $g(x) = 1$, and so forth. In the examples in the next sctions, we find $g(x)$ inductively. However, some graphs require more subtle techniques; see Exercise 5(c) for an example. The Sprague-Grundy function exists uniquely and is finite for all *progressively bounded* graphs. See Fraenkel (2000) for an efficient algorithm that computes it.

3.3 Examples.

1. Find the Sprague-Grundy function.

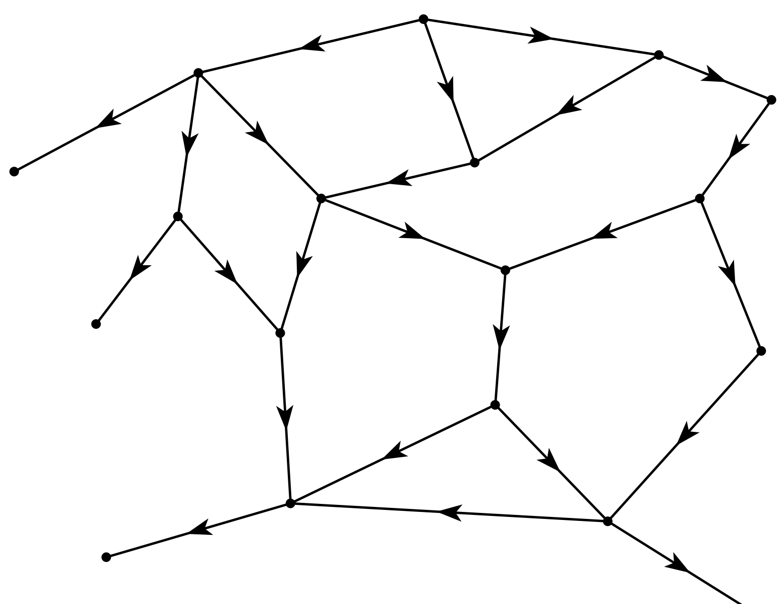


Fig. 3.2

2. What is the Sprague-Grundy function of the subtraction game with subtraction set $S = \{1, 2, 3\}$? The terminal vertex, 0, has SG-value 0. The vertex 1 can only be moved to 0 and $g(0) = 0$, so $g(1) = 1$. Similarly, 2 can move to 0 and 1 with $g(0) = 0$ and $g(1) = 1$, so $g(2) = 2$, and 3 can move to 0, 1 and 2, with $g(0) = 0$, $g(1) = 1$ and $g(2) = 2$, so $g(3) = 3$. But 4 can only move to 1, 2 and 3 with SG-values 1, 2 and 3, so $g(4) = 0$. Continuing in this way we see

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14...
$g(x)$	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2...

In general $g(x) = x \pmod 4$, i.e. $g(x)$ is the remainder when x is divided by 4.

3. *At-Least-Half.* Consider the one-pile game with the rule that you must remove at least half of the counters. The only terminal position is zero. We may compute the Sprague-Grundy function inductively as

x	0	1	2	3	4	5	6	7	8	9	10	11	12	...
$g(x)$	0	1	2	2	3	3	3	3	4	4	4	4	4	...

We see that $g(x)$ may be expressed as the exponent in the smallest power of 2 greater than x: $g(x) = \min\{k : 2^k > x\}$.

3.4 The Use of the Sprague-Grundy Function. Given the Sprague-Grundy function g of a graph, it is easy to analyze the corresponding graph game. Positions x for which $g(x) = 0$ are P-positions and all other positions are N-positions. The winning procedure is to choose at each move to move to a vertex with Sprague-Grundy value zero. This is easily seen by checking the conditions of Section 1.3:

(1) If x is a terminal position, $g(x) = 0$.

(2) At positions x for which $g(x) = 0$, every follower y of x is such that $g(y) \neq 0$, and

(3) At positions x for which $g(x) \neq 0$, there is at least one follower y such that $g(y) = 0$.

The Sprague-Grundy function thus contains a lot more information about a game than just the P- and N-positions. What is this extra information used for? As we will see in the Section 4, the Sprague-Grundy function allows us to analyze *sums* of graph games.

We may generalize the theory by replacing the hypothesis that the graph be progressively bounded by the hypothesis that the graph be **progressively finite**: *every path has a finite length.* This is essentially equivalent to the ending condition (6) of Section 1.2. Circuits would still be ruled out if we made such a change.

As an example of a graph that is progressively finite but not progressively bounded, consider the graph of the game in Figure 3.3 in which the first move is to choose the number of chips in a pile, and from then on to treat the pile as a nim pile. From the initial position each path has a finite length so the graph is progressively finite. But the graph is not progressively bounded since there is no upper limit to the length of a path from the initial position. The Sprague-Grundy theory can be extended to progressively finite graphs, but transfinite induction must be used. The SG-value of the initial position above would be the smallest ordinal greater than all integers, usually denoted by ω. We may also define nim positions with SG-values $\omega+1, \omega+2, \ldots, 2\omega, \ldots, \omega^2, \ldots, \omega^\omega$, etc., etc., etc. Except for Exercise 7, we do not pursue this topic further.

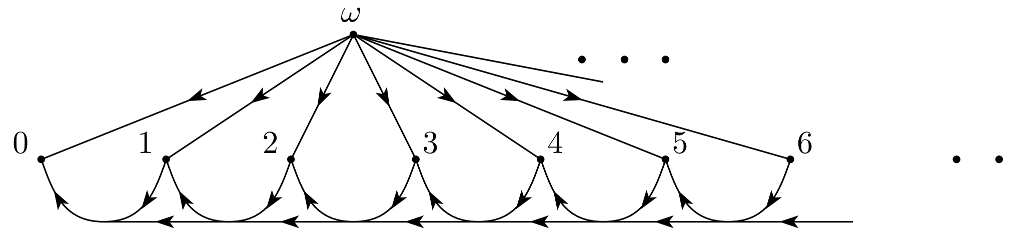


Fig 3.3 A progressively finite graph that is not progressively bounded.

3.5 Exercises.

1.

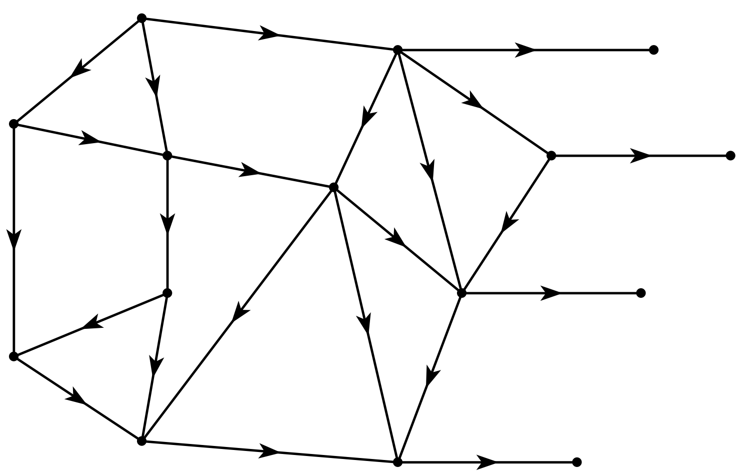


Fig 3.4 Find the Sprague-Grundy function.

2. Find the Sprague-Grundy function of the subtraction game with subtraction set $S = \{1, 3, 4\}$.

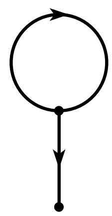
3. Consider the one-pile game with the rule that you may remove at most half the chips. Of course, you must remove at least one, so the terminal positions are 0 and 1. Find the Sprague-Grundy function.

4. (a) Consider the one-pile game with the rule that you may remove c chips from a pile of n chips if and only if c is a divisor of n, including 1 and n. For example, from a pile of 12 chips, you may remove 1, 2, 3, 4, 6, or 12 chips. The only terminal position is 0. Find the Sprague-Grundy function.

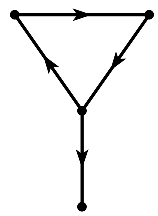
(b) Suppose the above rules are in force with the exception that it is not allowed to remove the whole pile. This is called the Aliquot game by Silverman, (1971). (See http://www.cut-the-knot.com/SimpleGames/Aliquot.html .) Thus, if there are 12 chips, you may remove 1, 2, 3, 4, or 6 chips. The only terminal position is 1. Find the Sprague-Grundy function.

5. The following directed graphs are not progressively finite. See if you can find the P- and N- positions and the Sprague-Grundy function.

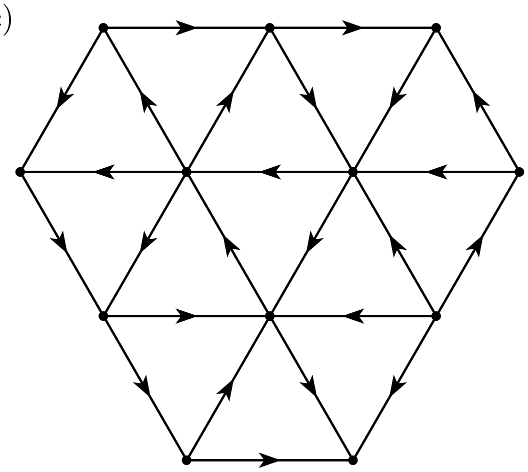
(a)



(b)



(c)

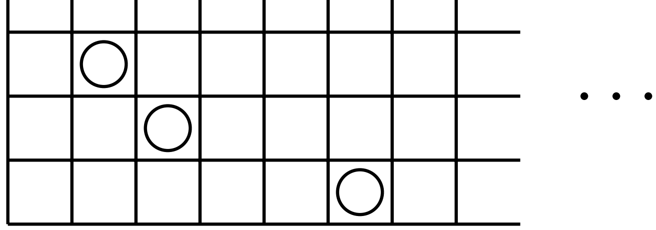


6. **Wythoff's Game.** (Wythoff (1907)) The positions of the Wythoff's game are given by a queen on a chessboard. Players, sitting on the same side of the board, take turns moving the queen. But the queen may only be moved vertically down, or horizontally to the left or diagonally down to the left. When the queen reaches the lower left corner, the game is over and the player to move last wins. Thinking of the squares of the board as vertices and the allowed moves of the queen as edges of a graph, this becomes a graph game. Find the Sprague-Grundy function of the graph by writing in each square of the 8 by 8 chessboard its Sprague-Grundy value. (You may play this game at http://www.chlond.demon.co.uk/Queen.html .)

7. **Two-Dimensional Nim** is played on a quarter-infinite board with a finite number of counters on the squares. A move consists in taking a counter and moving it any number of squares to the left on the same row, or moving it to any square whatever on any lower row. A square is allowed to contain any number of counters. If all the counters are on the lowest row, this is just the game Nimble of Exercise 2.3.

(a) Find the Sprague-Grundy values of the squares.

(b) After you learn the theory contained in the next section, come back and see if you can solve the position represented by the figure below. Is the position below a P-position or an N-position? If it is an N-position, what is a winning move? How many moves will this game last? Can it last an infinite number of moves?



4. Sums of Combinatorial Games.

Given several combinatorial games, one can form a new game played according to the following rules. A given initial position is set up in each of the games. Players alternate moves. A move for a player consists in selecting any one of the games and making a legal move in that game, leaving all other games untouched. Play continues until all of the games have reached a terminal position, when no more moves are possible. The player who made the last move is the winner.

The game formed by combining games in this manner is called the (**disjunctive**) **sum** of the given games. We first give the formal definition of a sum of games and then show how the Sprague-Grundy functions for the component games may be used to find the Sprague-Grundy function of the sum. This theory is due independently to R. P. Sprague (1935-6) and P. M. Grundy (1939).

4.1 The Sum of n Graph Games. Suppose we are given n progressively bounded graphs, $G_1 = (X_1, F_1), G_2 = (X_2, F_2), \ldots, G_n = (X_n, F_n)$. One can combine them into a new graph, $G = (X, F)$, called the **sum** of $G_1, G_2, \ldots, G_n$ and denoted by $G = G_1 + \cdots + G_n$ as follows. The set X of vertices is the Cartesian product, $X = X_1 \times \cdots \times X_n$. This is the set of all n-tuples $(x_1, \ldots, x_n)$ such that $x_i \in X_i$ for all i. For a vertex $x = (x_1, \ldots, x_n) \in X$, the set of followers of x is defined as

$$\begin{aligned} F(x) = F(x_1, \ldots, x_n) = & F_1(x_1) \times \{x_2\} \times \cdots \times \{x_n\} \\ & \cup \{x_1\} \times F_2(x_2) \times \cdots \times \{x_n\} \\ & \cup \cdots \\ & \cup \{x_1\} \times \{x_2\} \times \cdots \times F_n(x_n). \end{aligned}$$

Thus, a move from $x = (x_1, \ldots, x_n)$ consists in moving exactly one of the x_i to one of its followers (i.e. a point in $F_i(x_i)$). The graph game played on G is called the **sum** of the graph games $G_1, \ldots, G_n$.

If each of the graphs G_i is progressively bounded, then the sum G is progressively bounded as well. The maximum number of moves from a vertex $x = (x_1, \ldots, x_n)$ is the sum of the maximum numbers of moves in each of the component graphs.

As an example, the 3-pile game of nim may be considered as the sum of three one-pile games of nim. This shows that even if each component game is trivial, the sum may be complex.

4.2 The Sprague-Grundy Theorem. The following theorem gives a method for obtaining the Sprague-Grundy function for a sum of graph games when the Sprague-Grundy functions are known for the component games. This involves the notion of nim-sum defined earlier. The basic theorem for sums of graph games says that the Sprague-Grundy function of a sum of graph games is the nim-sum of the Sprague-Grundy functions of its component games. It may be considered a rather dramatic generalization of Theorem 1 of Bouton.

The proof is similar to the proof of Theorem 1.

Theorem 2. *If g_i is the Sprague-Grundy function of G_i, $i = 1, \ldots, n$, then $G = G_1 + \cdots + G_n$ has Sprague-Grundy function $g(x_1, \ldots, x_n) = g_1(x_1) \oplus \cdots \oplus g_n(x_n)$.*

Proof. Let $x = (x_1, \ldots, x_n)$ be an arbitrary point of X. Let $b = g_1(x_1) \oplus \cdots \oplus g_n(x_n)$. We are to show two things for the function $g(x_1, \ldots, x_n)$:

(1) For every non-negative integer $a < b$, there is a follower of $(x_1, \ldots, x_n)$ that has g-value a.

(2) No follower of $(x_1, \ldots, x_n)$ has g-value b.

Then the SG-value of x, being the smallest SG-value not assumed by one of its followers, must be b.

To show (1), let $d = a \oplus b$, and k be the number of digits in the binary expansion of d, so that $2^{k-1} \leq d < 2^k$ and d has a 1 in the kth position (from the right). Since $a < b$, b has a 1 in the kth position and a has a 0 there. Since $b = g_1(x_1) \oplus \cdots \oplus g_n(x_n)$, there is at least one x_i such that the binary expansion of $g_i(x_i)$ has a 1 in the kth position. Suppose for simplicity that $i = 1$. Then $d \oplus g_1(x_1) < g_1(x_1)$ so that there is a move from x_1 to some x_1' with $g_1(x_1') = d \oplus g_1(x_1)$. Then the move from $(x_1, x_2, \ldots, x_n)$ to $(x_1', x_2, \ldots, x_n)$ is a legal move in the sum, G, and

$$g_1(x_1') \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n) = d \oplus g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n) = d \oplus b = a.$$

Finally, to show (2), suppose to the contrary that $(x_1, \ldots, x_n)$ has a follower with the same g-value, and suppose without loss of generality that this involves a move in the first game. That is, we suppose that $(x_1', x_2, \ldots, x_n)$ is a follower of $(x_1, x_2, \ldots, x_n)$ and that $g_1(x_1') \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n)$. By the cancellation law, $g_1(x_1') = g_1(x_1)$. But this is a contradiction since no position can have a follower of the same SG-value. ■

One remarkable implication of this theorem is that every progressively bounded impartial game, when considered as a component in a sum of such games, behaves as if it were a nim pile. That is, it may be replaced by a nim pile of appropriate size (its Sprague-Grundy value) without changing the outcome, no matter what the other components of the sum may be. We express this observation by saying that *every (progressively bounded) impartial game is equivalent to some nim pile.*

4.3 Applications. 1. *Sums of Subtraction Games.* The one-pile subtraction game, $G(m)$, with subtraction set $S_m = \{1, 2, \ldots, m\}$, in which from 1 to m chips may be removed from the pile, has Sprague-Grundy function $g_m(x) = x \pmod{m+1}$, and $0 \leq g_m(x) \leq m$.

Consider the sum of three subtraction games. In the first game, $m = 3$ and the pile has 9 chips. In the second, $m = 5$ and the pile has 10 chips. And in the third, $m = 7$ and the pile has 14 chips. Thus, we are playing the game $G(3) + G(5) + G(7)$ and the initial position is $(9, 10, 14)$. The Sprague-Grundy value of this position is $g(9, 10, 14) = g_3(9) \oplus g_5(10) \oplus g_7(14) = 1 \oplus 4 \oplus 6 = 3$. One optimal move is to change the position in game $G(7)$ to have Sprague-Grundy value 5. This can be done by removing one chip from the pile of 14, leaving 13. There is another optimal move. Can you find it?

This shows the importance of knowing the Sprague-Grundy function. We present further examples of computing the Sprague-Grundy function for various one-pile games.

Note that although many of these one-pile games are trivial, as is one-pile nim, the Sprague-Grundy function has its main use in playing the sum of several such games.

2. *Even if Not All – All if Odd.* Consider the one-pile game with the rule that you can remove (1) any even number of chips provided it is not the whole pile, or (2) the whole pile provided it has an odd number of chips. There are two terminal positions, zero and two. We compute inductively,

x	0	1	2	3	4	5	6	7	8	9	10	...
$g(x)$	0	1	0	2	1	3	2	4	3	5	4	...

and we see that $g(2k) = k - 1$ and $g(2k - 1) = k$ for $k \geq 1$.

Suppose this game is played with three piles of sizes 10, 15 and 20. The SG-values are $g(10) = 4$, $g(15) = 7$ and $g(20) = 9$. Since $4 \oplus 7 \oplus 9 = 10$ is nit zero, this is an N-position. A winning move is to change the SG-value 9 to a 3. For this we may remove 12 chips from the pile of 20 leaving 8, since $g(8) = 3$.

3. *A Sum of Three Different Games.* Suppose you are playing a three pile take-away game. For the first pile of 18 chips, the rules of the previous game, Even if Not All – All if Odd, apply. For the second pile of 17 chips, the rules of At-Least-Half apply (Example 3.3.3). For the third pile of 7 chips, the rules of nim apply. First, we find the SG-values of the three piles to be 8, 5, and 7 respectively. This has nim-sum 10 and so is an N-position. It can be changed to a P-position by changing the SG-value of the first pile to 2. From the above table, this occurs for piles of 3 and 6 chips. We cannot move from 18 to 3, but we can move from 18 to 6. Thus an optimal move is to subtract 12 chips from the pile of 18 chips leaving 6 chips.

4.4 Take-and-Break Games. There are any other impartial combinatorial games that may be solved using the methods of this chapter. We describe Take-and-Break Games here, and in the next chapter, we look in greater depth at another impartial combinatorial game called Green Hackenbush. Take-and-Break Games are games where the rules allow taking and/or splitting one pile into two or more parts under certain conditions, thus increasing the number of piles.

1. *Lasker's Nim.* A generalization of Nim into a Take-and-Break Game is due to Emanuel Lasker, world chess champion from 1894 to 1921, and found in his book, *Brettspiele der Völker* (1931), 183-196.

Suppose that each player at his turn is allowed (1) to remove any number of chips from one pile as in nim, or (2) to split one pile containing at least two chips into two non-empty piles (no chips are removed).

Clearly the Sprague-Grundy function for the one-pile game satisfies $g(0) = 0$ and $g(1) = 1$. The followers of 2 are 0, 1 and $(1, 1)$, with respective Sprague-Grundy values of 0, 1, and $1 \oplus 1 = 0$. Hence, $g(2) = 2$. The followers of 3 are 0, 1, 2, and $(1, 2)$, with Sprague-Grundy values 0, 1, 2, and $1 \oplus 2 = 3$. Hence, $g(3) = 4$. Continuing in this manner, we see

x	0	1	2	3	4	5	6	7	8	9	10	11	...
$g(x)$	0	1	2	4	3	5	6	8	7	9	10	12	...

We therefore conjecture that $g(4k+1) = 4k+1$, $g(4k+2) = 4k+2$, $g(4k+3) = 4k+4$ and $g(4k+4) = 4k+3$, for all $k \ge 0$.

The validity of this conjecture may easily be verified by induction as follows.

(a) The followers of $4k+1$ that consist of a single pile have Sprague-Grundy values from 0 to $4k$. Those that consist of two piles, $(4k, 1), (4k-1, 2), \ldots, (2k+1, 2k)$, have even Sprague-Grundy values, and therefore $g(4k+1) = 4k+1$.

(b) The followers of $4k+2$ that consist of a single pile have Sprague-Grundy values from 0 to $4k+1$. Those that consist of two piles, $(4k+1, 1), (4k, 2), \ldots, (2k+1, 2k+1)$, have Sprague-Grundy values alternately divisible by 4 and odd, so that $g(4k+2) = 4k+2$.

(c) The followers of $4k+3$ that consist of a single pile have Sprague-Grundy values from 0 to $4k+2$. Those that consist of two piles, $(4k+2, 1), (4k+1, 2), \ldots, (2k+2, 2k+1)$, have odd Sprague-Grundy values, and in particular $g(4k+2, 1) = 4k+3$. Hence $g(4k+3) = 4k+4$.

(d) Finally, the followers of $4k+4$ that consist of a single pile have Sprague-Grundy values from 0 to $4k+2$, and $4k+4$. Those that consist of two piles, $(4k+3, 1)(4k+2, 2), \ldots, (2k+2, 2k+2)$, have Sprague-Grundy values alternately equal to 1 (mod 4) and even. Hence, $g(4k+4) = 4k+3$.

Suppose you are playing Lasker's nim with three piles of 2, 5, and 7 chips. What is your move? First, find the Sprague-Grundy value of the component positions to be 2, 5, and 8 respectively. The nim-sum of these three numbers is 15. You must change the position of Sprague-Grundy value 8 to a position of Sprague-Grundy value 7. This may be done by splitting the pile of 7 chips into two piles of say 1 and 6 chips. At the next move, your opponent will be faced with a four pile game of Lasker's nim with 1, 2, 5 and 6 chips. This has Sprague-Grundy value zero and so is a P-position.

2. *The Game of Kayles.* This game was introduced a century ago by Sam Loyd (see *Mathematical Puzzles of Sam Loyd, Vol 2.*, 1960, Dover Publications), and by H. E. Dudeney (see *The Canterbury Puzzles and Other Curious Problems*, 1958, Dover Publications, New York). Two bowlers face a line of 13 bowling pins in a row with the second pin already knocked down. "It is assumed that the ancient players had become so expert that they could always knock down any single kayle-pin, or any two kayle-pins that stood close together (i.e. adjacent pins). They therefore altered the game, and it was agreed that the player who knocked down the last pin was the winner."

This is one of our graph games played with piles of chips that can be described as follows. You may remove one or two chips from any pile after which, if desired, you may split that pile into two nonempty piles.

Removing one chip from a pile without splitting the pile corresponds to knocking down the end chip of a line. Removing one chip with splitting the pile into two parts corresponds to knocking down a pin in the interior of the line. Similarly for removing two chips.

Let us find the Sprague-Grundy function, $g(x)$, for this game. The only terminal position is a pile with no chips, so $g(0) = 0$. A pile on one chip can be moved only to an empty pile, so $g(1) = 1$. A pile of two chips can be moved either to a pile of one or zero chips, so $g(2) = 2$. A pile of three chips can be moved to a pile of two or one chips, (SG-value 1 and 2) or to two piles of one chip each (SG-value 0), so $g(3) = 3$. Continuing in this way, we find more of the Sprague-Grundy values in Table 4.1.

$y\backslash z$	0	1	2	3	4	5	6	7	8	9	10	11
0	**0**	1	2	**3**	1	4	**3**	2	1	**4**	2	**6**
12	4	1	2	**7**	1	4	**3**	2	1	**4**	**6**	7
24	4	1	2	8	**5**	4	**7**	2	1	8	6	7
36	4	1	2	**3**	1	4	7	2	1	8	2	7
48	4	1	2	8	1	4	7	2	1	**4**	2	7
60	4	1	2	8	1	4	7	2	1	8	**6**	7
72	4	1	2	8	1	4	7	2	1	8	2	7

Table 4.1. The SG-values for Kayles. Entries for the Table are for $g(y + z)$ where y is on the left side and z is at the top.

From $x = 72$ on, the SG-values are periodic with period 12, with the values in the last line repeating forever. There are only 14 exceptions to the sequence of the last line. They are displayed in bold face type. The last exception is at $x = 70$.

3. *Dawson's Chess.* One of T. R. Dawson's fanciful problems in *Caissa's Wild Roses* (1935), republished in *Five Classics of Fairy Chess* by Dover (1973), is give-away chess played with pawns. "Given two equal lines of opposing Pawns, White on 3rd rank, Black on 5th, n adjacent files, White to play at losing game, what is the result?" (Captures must be made, last player to move loses.) We treat this game here under the normal ending rule, that the last player to move wins.

Those unfamiliar with the movement of the pawn in chess might prefer a different way of describing the game as a kind of misére tic-tac-toe on a line of n squares, with both players using X as the mark. A player may place an X on any empty square provided it is not next to an X already placed. (The player who is forced to move next to another X loses.)

This game may be described as a game of removing chips from a pile and possibly splitting a pile into two piles. If $n = 1$ there is only one move to $n = 0$, ending the game. For $n > 1$, a move of placing an X at the end of a line removes the two squares at that end of the line from the game. This corresponds to removing two chips from the pile. Similarly, placing an X one square from the end corresponds to removing three chips from the pile. Placing an X in one of the squares not at the end or next to it corresponds to removing three chips from the pile and splitting the pile into to parts. Thus the rules of the game are: (1) You may remove one chip if it is the whole pile, or (2) you may remove two chips from any pile, or (3) you may remove three chips from any pile and if desired split that pile into two parts.

The Sprague-Grundy sequence begins 0, 1, 1, 2, 0, 3, 1, 1, 0, 3, 3, . . . It is eventually periodic with period 34. There are only 7 exceptions and the last exception occurs at $n = 51$.

4. *Grundy's Game.* The only legal move is to split a single pile into two nonempty piles of different sizes. Thus the only terminal positions are piles of size one or two. Is the Sprague Grundy sequence eventually periodic? This is unknown though 10,000,000 values have been computed as of 1996. (See R. K. Guy (1996).)

4.5 Exercises.

1. Consider the take-away game with the rule that (1) you may remove any even number of chips from any pile, or (2) you may remove any pile consisting of one chip. The only terminal position is 0. Find the Sprague-Grundy function.

2. Consider the one-pile game with the rule that you may remove (1) any number of chips divisible by three provided it is not the whole pile, or (2) the whole pile, but only if it has 2 (mod 3) chips (that is, only if it has 2, or 5, or 8, ... chips). The terminal positions are zero, one and three. Find the Sprague-Grundy function.

3. Suppose you are playing a three-pile subtraction game. For the first pile of 18 chips, the rules of Exercise 1 hold. For the second pile of 17 chips, the rules of Exercise 2 apply. For the third pile of 7 chips, the rules of nim apply. What is the Sprague-Grundy value of this position? Find an optimal move.

4. Solve the Kayles problem of Dudeney and Loyd. Of the 13 bowling pins in a row, the second has been knocked down, leaving:

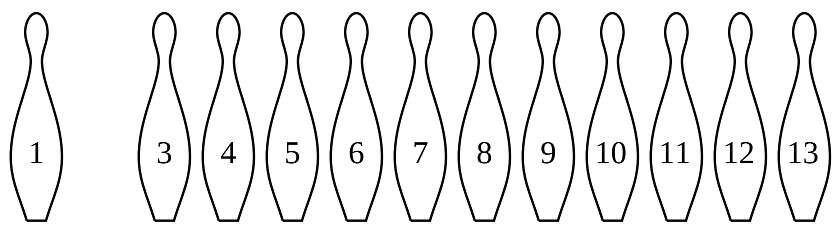


Figure 4.1 The Kayles problem of Dudeney and Loyd

(a) Show this is an N-position. You may use Table 4.1.
(b) Find a winning move. Which pin(s) should be knocked down?
(c) Now that you know the theory and have Table 4.1 at hand, you can go to http://www.chlond.demon.co.uk/Kayles.html and beat the computer.

5. Suppose at each turn a player may (1) remove one or two chips, or (2) remove one chip and split the remaining chips into two piles.
(a) Find the Sprague-Grundy function.
(b) Suppose the starting position consists of one pile with 15 chips. Find an optimal first move.

6. Suppose that at each turn a player may (1) remove one chip if it is a whole pile, or (2) remove two or more chips and, if desired, split the remaining chips into two piles. Find the Sprague-Grundy function.

7. Suppose that at each turn a player may select one pile and remove c chips if $c = 1$ (mod 3) and, if desired, split the remaining chips into two piles. Find the Sprague-Grundy function.

8. **Rims.** A position in the game of Rims is a finite set of dots in the plane, possibly separated by some nonintersecting closed loops. A move consists of drawing a closed loop passing through any positive number of dots (at least one) but not touching any other loop. Players alternate moves and the last to move wins.
(a) Show that this game is a disguised form of nim.
(b) In the position given in Figure 4.2, find a winning move, if any.

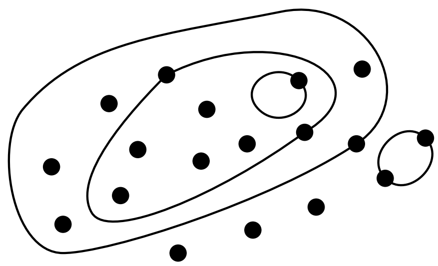


Figure 4.2 A Rims Position

9. **Rayles.** There are many geometric games like Rims treated in *Winning Ways*, Chapter 17. In one of them, called Rayles, the positions are those of Rims, but in Rayles, each closed loop must pass through exactly one or two points.
(a) Show that this game is a disguised form of Kayles.
(b) In the position given in Figure 4.2, find a winning move, if any.

10. **Grundy's Game.** (a) Compute the Sprague-Grundy function for Grundy's game, Example 3 Section 4.5, for a pile of n chips for $n = 1, 2, \ldots, 13$.
(b) In Grundy's game with three piles of sizes 5, 8, and 13, find all winning first moves, if any.

5. Green Hackenbush

The game of Hackenbush is played by hacking away edges from a rooted graph and removing those pieces of the graph that are no longer connected to the ground. A rooted graph is an undirected graph with every edge attached by some path to a special vertex called **the root** or **the ground**. The ground is denoted in the figures that follow by a dotted line.

In this section, we discuss the impartial version of this game in which either player at his turn may chop any edge. This version is called Green Hackenbush where each edge is considered to be colored green. There is also the partizan version of the game, called Blue-Red Hackenbush, in which some edges are colored blue and some are colored red. Player I may only chop the blue edges and Player II may only chop the red edges so the game is no longer impartial. Blue-Red Hackenbush is the first game treated in *Winning Ways*. In the general game of Hackenbush, there may be some blue edges available only to Player I, some red edges available only to Player II, and some green edges that either player may chop.

5.1 Bamboo Stalks. As an introduction to the game of Green Hackenbush, we investigate the case where the graph consists of a number of bamboo stalks as in the left side of Figure 5.1. A bamboo stalk with n segments is a linear graph of n edges with the bottom of the n edges rooted to the ground. A move consists of hacking away one of the segments, and removing that segment and all segments above it no longer connected to the ground. Players alternate moves and the last player to move wins. A single bamboo stalk of n segments can be moved into a bamboo stalk of any smaller number of segments from $n-1$ to 0. So a single bamboo stalk of n segments is equivalent to a nim pile of n chips. Playing a sum of games of bamboo stalks is thus equivalent to playing nim.

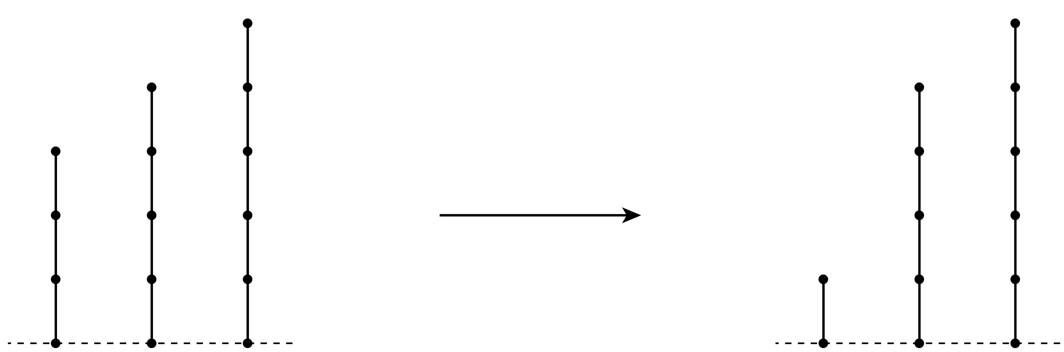


Figure 5.1

For example, the "forest" of three stalks on the left, is equivalent to the game of nim with three piles of 3, 4 and 5 chips. Since $3 \oplus 4 \oplus 5 = 2$, this is an N-position which can be moved to a P-position by hacking the second segment of the stalk with three segments, leaving a stalk of one segment. The resulting position on the right has Sprague-Grundy value 0, and so is a P-position.

5.2 Green Hackenbush on Trees. Played with bamboo stalks, Green Hackenbush is just nim in a rather transparent disguise. But what happens if we allow more general structures than these simple bamboo stalks? Suppose we have the "forest" of three rooted trees found in Figure 5.2. A "rooted tree" is a graph with a distinguished vertex called the root, with the property that from every vertex there is a unique path to the root. Essentially this means there are no circuits.

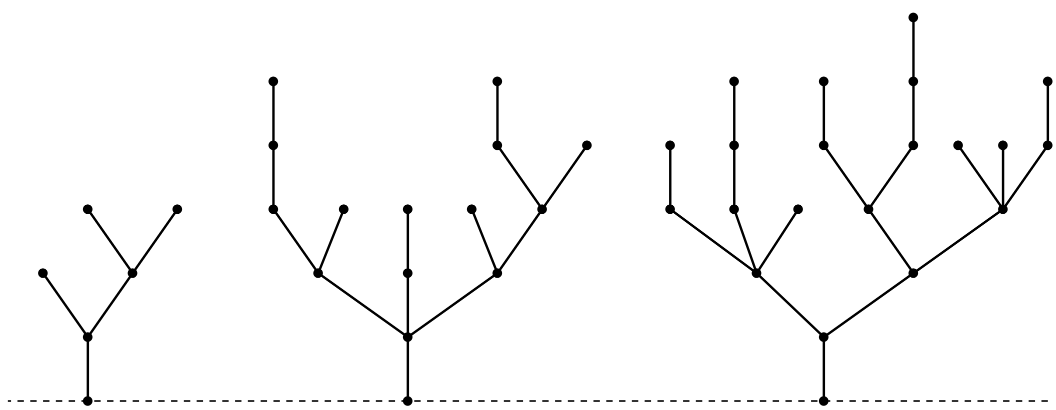


Figure 5.2

Again a move consists of hacking away any segment and removing that segment and anything not connected to the ground. Since the game is impartial, the general theory of Section 4 tells us that each such tree is equivalent to some nim pile, or if you will, to some bamboo stalk. The problem is to find the Sprague-Grundy values of each of the trees.

This may be done using the following principle, known in its more general form as the **Colon Principle:** *When branches come together at a vertex, one may replace the branches by a non-branching stalk of length equal to their nim sum.*

Let us see how this principle works to find the bamboo stalk equivalent to the left tree of Figure 5.2. There are two vertices with two branches. The higher of these vertices has two branches each with one edge. The nim sum of one and one is zero; so the two branches may be replaced by a branch with zero edges. That is to say, the two branches may be removed. This leaves a Y-shaped tree and the same reasoning may be used to show that the two branches at the Y may be removed. Thus the tree on the left of Figure 5.2 is equivalent to a nim pile of size one.

This may have been a little too simple for illustrative purposes, so let's consider the second tree of Figure 5.2. The leftmost branching vertex has two branches of lengths three and one. The nim sum of three and one is two, so the two branches may be replaced by a single branch of length two. The rightmost branching vertex has two branches of lengths one and two whose nim sum is three, so the two branches may be replaced by a single branch of length three. See the reduction in Figure 5.3. Continuing in like manner we arrive at the conclusion that the second tree of Figure 5.2 is equivalent to a nim pile of 8 chips.

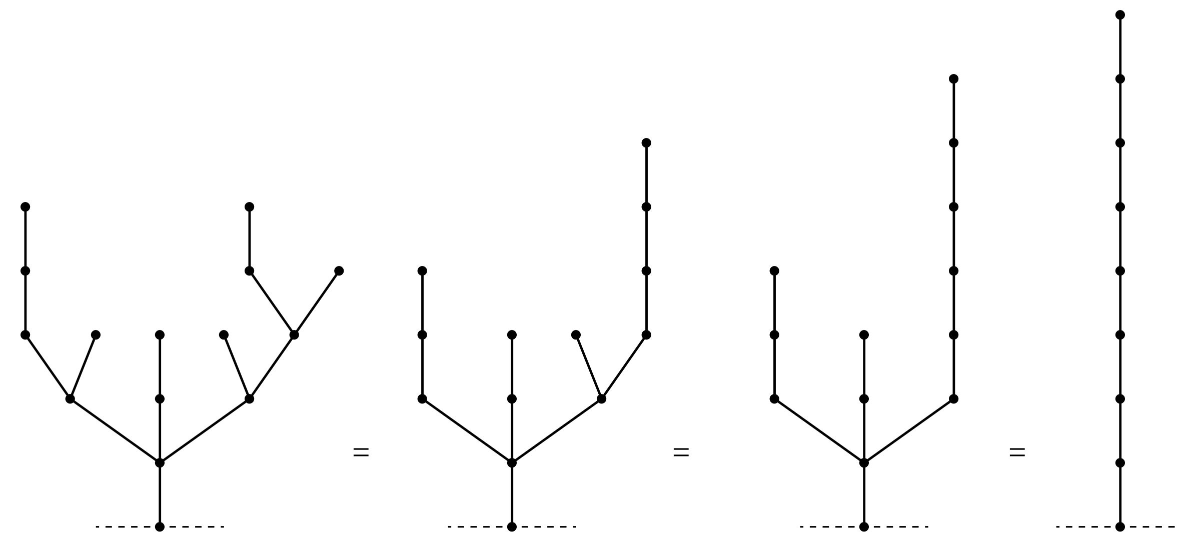


Figure 5.3

Now try your luck with the third tree of Figure 5.2. See if you can show that it is equivalent to a nim pile of size 4.

Now we can compute the Sprague-Grundy value of the sum of the three trees of Figure 5.2. It is $1 \oplus 8 \oplus 4 = 13$. Since this is not zero, it is a win for the next person to play. The next problem is to find a winning move. It is clear that there is a winning move using the second tree that may be obtained by chopping some edge to arrive at a tree of Sprague-Grundy value 5. But which edge must be chopped to achieve this?

The last version of the tree in Figure 5.3 has length 8 because the three branches of the previous tree were 3, 2 and 6, whose nim-sum is $3 \oplus 2 \oplus 6 = 7$. To achieve length 5 in the last tree, we must change one of the three branches to achieve nim-sum 4. This may be done most easily by chopping the leftmost branch entirely, since $2 \oplus 6 = 4$. Alternatively, we may hack away the top edge of the middle branch leaving one edge, because $3 \oplus 1 \oplus 6 = 4$.

Each of these moves is easily translated into the corresponding chop on the tree on the left of Figure 5.3. However, there is another way to reduce this tree to Sprague-Grundy value 5, that uses the right branch of the tree. See if you can find it.

The method of pruning trees given by the colon principle works to reduce all trees to a single bamboo stalk. One starts at the highest branches first, and then applies the principle inductively down to the root. We now show the validity of this principle for rooted graphs that may have circuits and several edges attached to the ground.

Proof of the Colon Principle. Consider a fixed but arbitrary graph, G, and select an arbitrary vertex, x, in G. Let H_1 and H_2 be arbitrary trees (or graphs) that have the same Sprague-Grundy value. Consider the two graphs $G_1 = G_x : H_1$ and $G_2 = G_x : H_2$, where $G_x : H_i$ represents the graph constructed by attaching the tree H_i to the vertex x of the graph G. The colon principle states that the two graphs G_1 and G_2 have the same Sprague-Grundy value. Consider the sum of the two games as in Figure 5.4.

The claim that G_1 and G_2 have the same Sprague-Grundy value is equivalent to the claim that the sum of the two games has Sprague-Grundy value 0. In other words, we are to show that the sum $G_1 + G_2$ is a P-position.

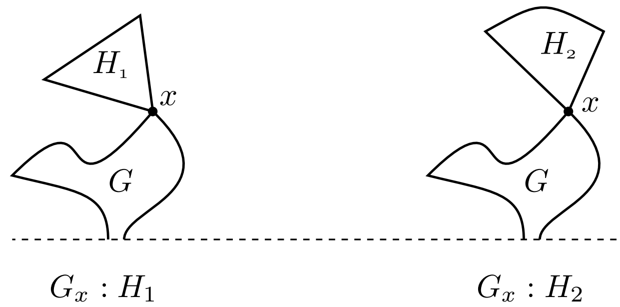


Figure 5.4

Here is a strategy that guarantees you a win if you are the second player to move in $G_1 + G_2$. If the first player moves by chopping one of the edges in G in one of the games, then you chop the same edge in G in the other game. (Such a pair of moves may delete H_1 and H_2 from the games, but otherwise H_1 and H_2 are not disturbed.) If the first player moves by chopping an edge in H_1 or H_2, then the Sprague-Grundy values of H_1 and H_2 are no longer equal, so that there exists a move in H_1 or H_2 that keeps the Sprague-Grundy values the same. In this way you will always have a reply to every move he may make. This means you will make the last move and so win. ■

5.3 Green Hackenbush on general rooted graphs. We now consider arbitrary graphs. These graphs may have circuits and loops and several segments may be attached to the ground. Consider Figure 5.5 as an example.

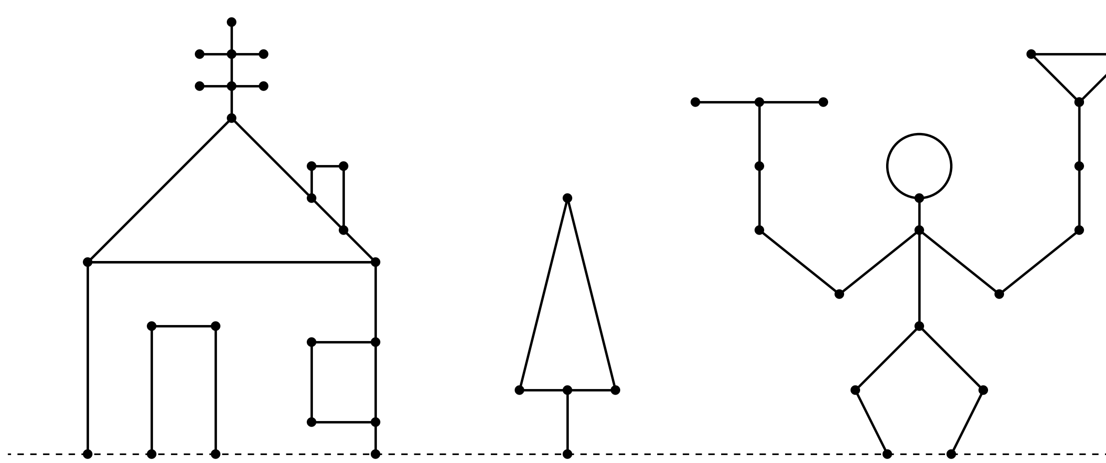


Figure 5.5

From the general theory of Chapter 4, each separate graph is equivalent to some nim pile. To find the equivalent nim pile, we look for an equivalent tree, from which the equivalent nim pile may be found. This is done using the **fusion principle**. We **fuse** two neighboring vertices by bringing them together into a single vertex and bending the edge joining them into a loop. A loop is an edge joining a vertex to itself, as for example the head of the juggler on the right of Figure 5.5. As far as Green Hackenbush is concerned, a loop may be replaced by a leaf (an edge with one end unattached).

The Fusion Principle: *The vertices on any circuit may be fused without changing the Sprague-Grundy value of the graph.*

The fusion principle allows us to reduce an arbitrary rooted graph into an equivalent tree which can be further reduced to a nim pile by the colon principle. Let us see how this works on the examples of Figure 5.5.

Consider the door in the house on the left. The two vertices on the ground are the same vertex (remember the ground is really a single vertex) so the door is really the same as a triangle with one vertex attached to the ground. The fusion principle says that this is equivalent to a single vertex with three loops attached. Each loop is equivalent to a nim pile of size 1, and the nim sum of these is also a nim pile of size 1.

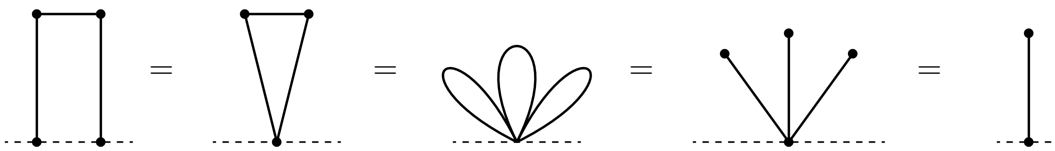


Figure 5.6

We see more generally that a circuit with an odd number of edges reduces to a single edge, and a circuit with an even number of edges reduces to a single vertex. For example, the circuit of four edges in the Christmas tree in the center of Figure 5.5 reduces to a single vertex with four loops, which reduces to a single vertex. So the Christmas tree is equivalent to a nim pile of size 1. Similarly, the chimney on the house becomes a single vertex, and the window on the right of the house is also a single vertex. Continuing further, we find that the house (door included) has SG-value 3.

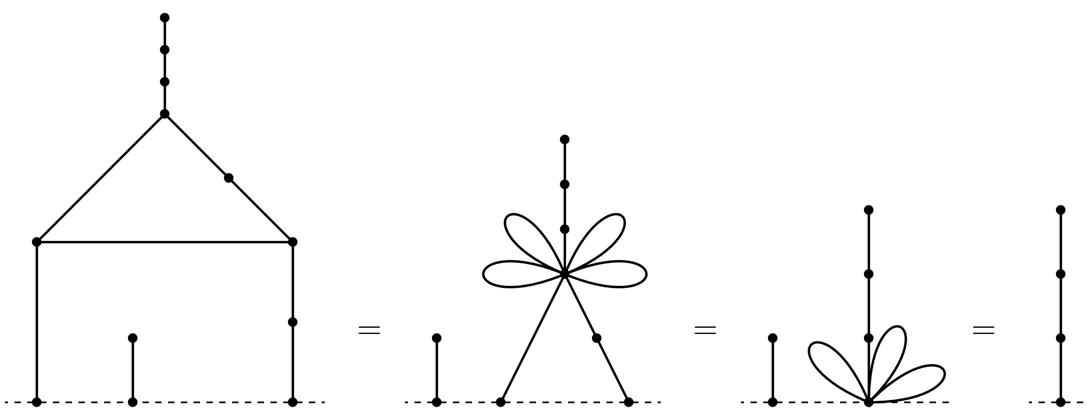


Figure 5.7

Now see if you can show that the juggler on the right of Figure 5.5 has SG-value 4. And then see if you can find a winning move in the Hackenbush position given by Figure 5.5.

The proof of the fusion principle is somewhat longer than the proof of the colon principle, and so is omitted. For a proof, see *Winning Ways*, Chapter 7.

5.4 Exercises.

1. (Stolen from *Fair Game* by Richard Guy.) Find the SG-values of the graphs in Figure 5.8, and find a winning move, if any.

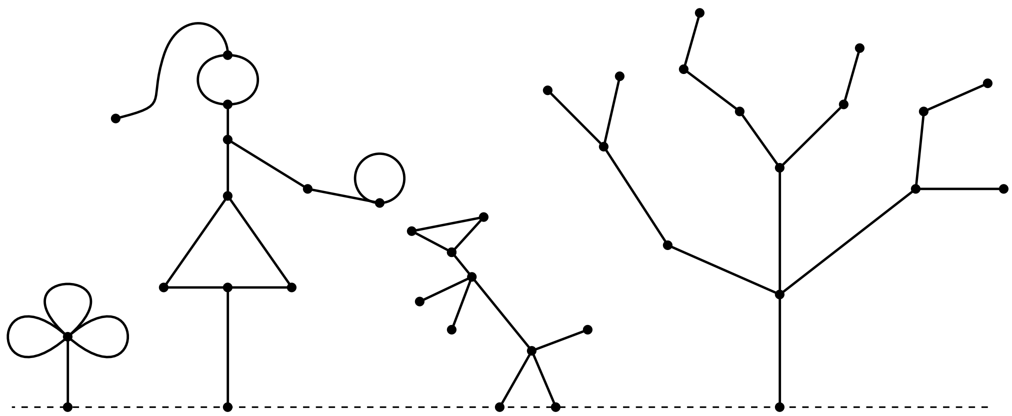


Figure 5.8

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