GENERATING FUNCTIONS AND RECURRENCE RELATIONS
Recurrence Relations

Suppose $a_0, a_1, a_2, \ldots, a_n, \ldots,$ is an infinite sequence. A recurrence relation is a set of equations

$$a_n = f_n(a_{n-1}, a_{n-2}, \ldots, a_{n-k}). \quad (1)$$

The whole sequence is determined by (6) and the values of $a_0, a_1, \ldots, a_{k-1}$. 

Linear Recurrence

Fibonacci Sequence

\[ a_n = a_{n-1} + a_{n-2} \quad n \geq 2. \]

\[ a_0 = a_1 = 1. \]
\[ b_n = |B_n| = |\{x \in \{a, b, c\}^n : aa \text{ does not occur in } x\}|. \]

\[ b_1 = 3 : a \ b \ c \]

\[ b_2 = 8 : ab \ ac \ ba \ bb \ bc \ ca \ cb \ cc \]

\[ b_n = 2b_{n-1} + 2b_{n-2} \quad n \geq 2. \]
\[ b_n = 2b_{n-1} + 2b_{n-2} \quad n \geq 2. \]

Let
\[ B_n = B_n^{(b)} \cup B_n^{(c)} \cup B_n^{(a)} \]
where \( B_n^{(\alpha)} = \{ x \in B_n : x_1 = \alpha \} \) for \( \alpha = a, b, c \).

Now \( |B_n^{(b)}| = |B_n^{(c)}| = |B_{n-1}| \). The map \( f : B_n^{(b)} \to B_{n-1} \),
\[ f(bx_2x_3 \ldots x_n) = x_2x_3 \ldots x_n \] is a bijection.

\[ B_n^{(a)} = \{ x \in B_n : x_1 = a \text{ and } x_2 = b \text{ or } c \} \]. The map \( g : B_n^{(a)} \to B_{n-1}^{(b)} \cup B_{n-1}^{(c)} \),
\[ g(ax_2x_3 \ldots x_n) = x_2x_3 \ldots x_n \] is a bijection.

Hence, \( |B_n^{(a)}| = 2|B_{n-2}| \).
H_n is the minimum number of moves needed to shift n rings from Peg 1 to Peg 2. One is not allowed to place a larger ring on top of a smaller ring.
Generating Functions
We see that $H_1 = 1$ and $H_n = 2H_{n-1} + 1$ for $n \geq 2$.

So,

$$\frac{H_n}{2^n} - \frac{H_{n-1}}{2^{n-1}} = \frac{1}{2^n}.$$  

Summing these equations give

$$\frac{H_n}{2^n} - \frac{H_1}{2} = \frac{1}{2^n} + \frac{1}{2^{n-1}} + \cdots + \frac{1}{4} = \frac{1}{2} - \frac{1}{2^n}.$$  

So

$$H_n = 2^n - 1.$$
$A$ has $n$ dollars. Everyday $A$ buys one of a Bun (1 dollar), an Ice-Cream (2 dollars) or a Pastry (2 dollars). How many ways are there (sequences) for $A$ to spend his money? Ex. BBPIIPBI represents “Day 1, buy Bun. Day 2, buy Bun etc.”.

\[ u_n = \text{number of ways} \]
\[ = u_{n,B} + u_{n,I} + u_{n,P} \]

where $u_{n,B}$ is the number of ways where $A$ buys a Bun on day 1 etc.

\[ u_{n,B} = u_{n-1}, \quad u_{n,I} = u_{n,P} = u_{n-2}. \]

So

\[ u_n = u_{n-1} + 2u_{n-2}, \]

and

\[ u_0 = u_1 = 1. \]
If \( a_0, a_1, \ldots, a_n \) is a sequence of real numbers then its (ordinary) generating function \( a(x) \) is given by

\[
a(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + \cdots
\]

and we write

\[
a_n = [x^n]a(x).
\]

For more on this subject see Generatingfunctionology by the late Herbert S. Wilf. The book is available from
https://www.math.upenn.edu// wilf/DownlIdGF.html
\[ a_n = 1 \]

\[ a(x) = \frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^n + \cdots \]

\[ a_n = n + 1. \]

\[ a(x) = \frac{1}{(1 - x)^2} = 1 + 2x + 3x^2 + \cdots + (n + 1)x^n + \cdots \]

\[ a_n = n. \]

\[ a(x) = \frac{x}{(1 - x)^2} = x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots \]
Generalised binomial theorem:

\[ a_n = \binom{\alpha}{n} \]

\[ a(x) = (1 + x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n. \]

where

\[ \binom{\alpha}{n} = \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - n + 1)}{n!}. \]

\[ a_n = \binom{m+n-1}{n} \]

\[ a(x) = \frac{1}{(1 - x)^m} = \sum_{n=0}^{\infty} \binom{-m}{n} (-x)^n = \sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n. \]
Given a recurrence relation for the sequence \((a_n)\), we

(a) Deduce from it, an equation satisfied by the generating function \(a(x) = \sum_n a_n x^n\).

(b) Solve this equation to get an explicit expression for the generating function.

(c) Extract the coefficient \(a_n\) of \(x^n\) from \(a(x)\), by expanding \(a(x)\) as a power series.
Solution of linear recurrences

\[ a_n - 6a_{n-1} + 9a_{n-2} = 0 \quad n \geq 2. \]

\[ a_0 = 1, \quad a_1 = 9. \]

\[ \sum_{n=2}^{\infty} (a_n - 6a_{n-1} + 9a_{n-2})x^n = 0. \] (2)
\[
\sum_{n=2}^{\infty} a_n x^n = a(x) - a_0 - a_1 x
\]
\[
= a(x) - 1 - 9x.
\]
\[
\sum_{n=2}^{\infty} 6a_{n-1} x^n = 6x \sum_{n=2}^{\infty} a_{n-1} x^{n-1}
\]
\[
= 6x(a(x) - a_0)
\]
\[
= 6x(a(x) - 1).
\]
\[
\sum_{n=2}^{\infty} 9a_{n-2} x^n = 9x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}
\]
\[
= 9x^2 a(x).
\]

Generating Functions
\[
a(x) - 1 - 9x - 6x(a(x) - 1) + 9x^2 a(x) = 0
\]
or
\[
a(x)(1 - 6x + 9x^2) - (1 + 3x) = 0.
\]

\[
a(x) = \frac{1 + 3x}{1 - 6x + 9x^2} = \frac{1 + 3x}{(1 - 3x)^2}
\]

\[
= \sum_{n=0}^{\infty} (n+1)3^n x^n + 3x \sum_{n=0}^{\infty} (n+1)3^n x^n
\]

\[
= \sum_{n=0}^{\infty} (n+1)3^n x^n + \sum_{n=0}^{\infty} n3^n x^n
\]

\[
= \sum_{n=0}^{\infty} (2n + 1)3^n x^n.
\]

\[
a_n = (2n + 1)3^n.
\]
Fibonacci sequence:

\[ \sum_{n=2}^{\infty} (a_n - a_{n-1} - a_{n-2})x^n = 0. \]

\[ \sum_{n=2}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0. \]

\[ (a(x) - a_0 - a_1 x) - (x(a(x) - a_0)) - x^2 a(x) = 0. \]

\[ a(x) = \frac{1}{1 - x - x^2}. \]
\[ a(x) = \frac{1}{(\xi_1 - x)(\xi_2 - x)} \]

\[ = \frac{1}{\xi_1 - \xi_2} \left( \frac{1}{\xi_1 - x} - \frac{1}{\xi_2 - x} \right) \]

\[ = \frac{1}{\xi_1 - \xi_2} \left( \frac{\xi_1^{-1}}{1 - x/\xi_1} - \frac{\xi_2^{-1}}{1 - x/\xi_2} \right) \]

where

\[ \xi_1 = -\frac{\sqrt{5} + 1}{2} \quad \text{and} \quad \xi_2 = \frac{\sqrt{5} - 1}{2} \]

are the 2 roots of

\[ x^2 + x - 1 = 0. \]
Therefore,

\[
a(x) = \frac{\xi_1^{-1}}{\xi_1 - \xi_2} \sum_{n=0}^{\infty} \xi_1^{-n} x^n - \frac{\xi_2^{-1}}{\xi_1 - \xi_2} \sum_{n=0}^{\infty} \xi_2^{-n} x^n
\]

\[
= \sum_{n=0}^{\infty} \frac{\xi_1^{1-n} - \xi_2^{1-n}}{\xi_1 - \xi_2} x^n
\]

and so

\[
a_n = \frac{\xi_1^{1-n} - \xi_2^{1-n}}{\xi_1 - \xi_2}
\]

\[
= \frac{1}{\sqrt{5}} \left( \left( \frac{\sqrt{5} + 1}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right).
\]
Inhomogeneous problem

\[ a_n - 3a_{n-1} = n^2 \quad n \geq 1. \]

\[ a_0 = 1. \]

\[
\sum_{n=1}^{\infty} (a_n - 3a_{n-1})x^n = \sum_{n=1}^{\infty} n^2 x^n
\]

\[
\sum_{n=1}^{\infty} n^2 x^n = \sum_{n=2}^{\infty} n(n-1)x^n + \sum_{n=1}^{\infty} nx^n
\]

\[
= \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}
\]

\[
= \frac{x + x^2}{(1-x)^3}
\]

\[
\sum_{n=1}^{\infty} (a_n - 3a_{n-1})x^n = a(x) - 1 - 3xa(x)
\]

\[
= a(x)(1 - 3x) - 1.
\]
\[ a(x) = \frac{x + x^2}{(1 - x)^3(1 - 3x)} + \frac{1}{1 - 3x} \]

\[ = \frac{A}{1 - x} + \frac{B}{(1 - x)^2} + \frac{C}{(1 - x)^3} + \frac{D + 1}{1 - 3x} \]

where

\[ x + x^2 \equiv A(1 - x)^2(1 - 3x) + B(1 - x)(1 - 3x) + C(1 - 3x) + D(1 - x)^3. \]

Then

\[ A = -1/2, \quad B = 0, \quad C = -1, \quad D = 3/2. \]

Generating Functions
So

\[ a(x) = \frac{-1/2}{1-x} - \frac{1}{(1-x)^3} + \frac{5/2}{1-3x} \]

\[ = \frac{-1}{2} \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} \binom{n+2}{2} x^n + \frac{5}{2} \sum_{n=0}^{\infty} 3^n x^n \]

So

\[ a_n = \frac{-1}{2} - \binom{n+2}{2} + \frac{5}{2} 3^n \]

\[ = \frac{-3}{2} - \frac{3n}{2} - \frac{n^2}{2} + \frac{5}{2} 3^n. \]
General case of linear recurrence

\[ a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = u_n, \quad n \geq k. \]

\( u_0, u_1, \ldots, u_{k-1} \) are given.

\[ \sum (a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} - u_n) x^n = 0 \]

It follows that for some polynomial \( r(x) \),

\[ a(x) = \frac{u(x) + r(x)}{q(x)} \]

where

\[ q(x) = 1 + c_1 x + c_2 x^2 + \cdots + c_k x^k = \prod_{i=1}^{k} (1 - \alpha_i x) \]

and \( \alpha_1, \alpha_2, \ldots, \alpha_k \) are the roots of \( p(x) = 0 \) where

\[ p(x) = x^k q(1/x) = x^k + c_1 x^{k-1} + \cdots + c_0. \]
Products of generating functions

\[ a(x) = \sum_{n=0}^{\infty} a_n x^n, \quad b(x)) = \sum_{n=0}^{\infty} b_n x^n. \]

\[ a(x)b(x) = (a_0 + a_1 x + a_2 x^2 + \cdots) \times (b_0 + b_1 x + b_2 x^2 + \cdots) \]
\[ = a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots \]
\[ = \sum_{n=0}^{\infty} c_n x^n \]

where

\[ c_n = \sum_{k=0}^{n} a_k b_{n-k}. \]
Derangements

\[ n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}. \]

**Explanation:** \( \binom{n}{k} d_{n-k} \) is the number of permutations with exactly \( k \) cycles of length 1. Choose \( k \) elements (\( \binom{n}{k} \) ways) for which \( \pi(i) = i \) and then choose a derangement of the remaining \( n - k \) elements.

So

\[ 1 = \sum_{k=0}^{n} \frac{1}{k! (n - k)!} d_{n-k} \]

\[ \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{1}{k! (n - k)!} d_{n-k} \right) x^n. \quad (3) \]
Let

\[ d(x) = \sum_{m=0}^{\infty} \frac{d_m}{m!} x^m. \]

From (3) we have

\[ \frac{1}{1 - x} = e^x d(x) \]

\[ d(x) = \frac{e^{-x}}{1 - x} \]

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \frac{(-1)^k}{k!} \right) x^n. \]

So

\[ \frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \]
Triangulation of \( n \)-gon

Let

\[
a_n = \text{number of triangulations of } P_{n+1}
= \sum_{k=0}^{n} a_k a_{n-k} \quad n \geq 2
\]

\( a_0 = 0, \ a_1 = a_2 = 1. \)
Explanation of (4):

\(a_k a_{n-k}\) counts the number of triangulations in which edge 
1, \(n + 1\) is contained in triangle 1, \(k + 1\), \(n + 1\).

There are \(a_k\) ways of triangulating 1, 2, \ldots, \(k + 1\), 1 and for 
each such there are \(a_{n-k}\) ways of triangulating 
\(k + 1\), \(k + 2\), \ldots, \(n + 1\), \(k + 1\).
\[ x + \sum_{n=2}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} \left( \sum_{k=0}^{n} a_k a_{n-k} \right) x^n. \]

But,
\[ x + \sum_{n=2}^{\infty} a_n x^n = a(x) \]

since \( a_0 = 0, a_1 = 1. \)

\[ \sum_{n=2}^{\infty} \left( \sum_{k=0}^{n} a_k a_{n-k} \right) x^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k a_{n-k} \right) x^n = a(x)^2. \]
So

\[ a(x) = x + a(x)^2 \]

and hence

\[ a(x) = \frac{1 + \sqrt{1 - 4x}}{2} \quad \text{or} \quad \frac{1 - \sqrt{1 - 4x}}{2}. \]

But \( a(0) = 0 \) and so

\[
\begin{align*}
    a(x) &= \frac{1 - \sqrt{1 - 4x}}{2} \\
    &= \frac{1}{2} - \frac{1}{2} \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^{2n-1}} \binom{2n-2}{n-1} (-4x)^n \right) \\
    &= \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n.
\end{align*}
\]

So

\[ a_n = \frac{1}{n} \binom{2n-2}{n-1}. \]
Exponential Generating Functions

Given a sequence $a_n, n \geq 0$, its exponential generating function (e.g.f.) $a_e(x)$ is given by

$$a_e(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

$a_n = 1, n \geq 0$ implies $a_e(x) = e^x$.

$a_n = n!, n \geq 0$ implies $a_e(x) = \frac{1}{1 - x}$.
Products of Exponential Generating Functions

Let \( a_e(x), b_e(x) \) be the e.g.f.’s respectively for \((a_n), (b_n)\) respectively. Then

\[
c_e(x) = a_e(x)b_e(x) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} \right) x^n
\]

\[
= \sum_{k=0}^{n} \frac{c_n}{n!} x^n
\]

where

\[
c_n = \binom{n}{k} a_k b_{n-k}.
\]
Interpretation

Suppose that we have a collection of labelled objects and each object has a “size” $k$, where $k$ is a non-negative integer. Each object is labelled by a set of size $k$. Suppose that the number of labelled objects of size $k$ is $a_k$.

Examples:
(a): Each object is a directed path with $k$ vertices and its vertices are labelled by $1, 2, \ldots, k$ in some order. Thus $a_k = k!$.
(b): Each object is a directed cycle with $k$ vertices and its vertices are labelled by $1, 2, \ldots, k$ in some order. Thus $a_k = (k - 1)!$. 

Generating Functions
Now take example (a) and let \( a_e(x) = \frac{1}{1-x} \) be the e.g.f. of this family. Now consider

\[
c_e(x) = a_e(x)^2 = \sum_{n=0}^{\infty} (n+1)x^n \text{ with } c_n = (n+1) \times n!.
\]

\( c_n \) is the number of ways of choosing an object of weight \( k \) and another object of weight \( n - k \) and a partition of \([n]\) into two sets \( A_1, A_2 \) of size \( k \) and labelling the first object with \( A_1 \) and the second with \( A_2 \).

Here \((n+1) \times n!\) represents taking a permutation and choosing \(0 \leq k \leq n\) and putting the first \( k \) labels onto the first path and the second \( n - k \) labels onto the second path.
We will now use this machinery to count the number $s_n$ of permutations that have an even number of cycles all of which have odd lengths:

**Cycles of a permutation**

Let $\pi : D \rightarrow D$ be a permutation of the finite set $D$. Consider the digraph $\Gamma_\pi = (D, A)$ where $A = \{(i, \pi(i)) : i \in D\}$. $\Gamma_\pi$ is a collection of vertex disjoint cycles. Each $x \in D$ being on a unique cycle. Here a cycle can consist of a loop i.e. when $\pi(x) = x$.


<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<th>7</th>
<th>8</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\pi(i)$</td>
<td>6</td>
<td>2</td>
<td>7</td>
<td>10</td>
<td>3</td>
<td>8</td>
<td>9</td>
<td>1</td>
<td>5</td>
<td>4</td>
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The cycles are $(1, 6, 8), (2), (3, 7, 9, 5), (4, 10)$.
In general consider the sequence $i, \pi(i), \pi^2(i), \ldots$.

Since $D$ is finite, there exists a first pair $k < \ell$ such that $\pi^k(i) = \pi^\ell(i)$. Now we must have $k = 0$, since otherwise putting $x = \pi^{k-1}(i) \neq y = \pi^{\ell-1}(i)$ we see that $\pi(x) = \pi(y)$, contradicting the fact that $\pi$ is a permutation.

So $i$ lies on the cycle $C = (i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i), i)$.

If $j$ is not a vertex of $C$ then $\pi(j)$ is not on $C$ and so we can repeat the argument to show that the rest of $D$ is partitioned into cycles.
Now consider

\[ a_e(x) = \sum_{m=0}^{\infty} \frac{(2m)!}{(2m+1)!} x^{2m+1} \]

Here

\[ a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ (n-1)! & \text{if } n \text{ is odd} \end{cases} \]

Thus each object is an odd length cycle \( C \), labelled by \( |C| \).

Note that

\[ a_e(x) = \left( x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \right) - \left( \frac{x^2}{2} + \frac{x^4}{4} + \cdots \right) \]

\[ = \log \left( \frac{1}{1-x} \right) - \frac{1}{2} \log \left( \frac{1}{1-x^2} \right) \]

\[ = \log \sqrt{\frac{1+x}{1-x}} \]
Now consider $a_e(x)^\ell$. The coefficient of $x^n$ in this series is $\frac{c_n}{n!}$ where $c_n$ is the number of ways of choosing an ordered sequence of $\ell$ cycles of lengths $a_1, a_2, \ldots, a_\ell$ where $a_1 + a_2 + \cdots + a_\ell = n$. And then a partition of $[n]$ into $A_1, A_2, \ldots, A_\ell$ where $|A_i| = a_i$ for $i = 1, 2, \ldots, \ell$. And then labelling the $i$th cycle with $A_i$ for $i = 1, 2, \ldots, \ell$.

We looked carefully at the case $\ell = 2$ and this needs a simple inductive step.

It follows that the coefficient of $x^n$ in \( \frac{a_e(x)^\ell}{\ell!} \) is $\frac{c_n}{n!}$ where $c_n$ is the number of ways of choosing a set (unordered sequence) of $\ell$ cycles of lengths $a_1, a_2, \ldots, a_\ell$ . . .

What we therefore want is the coefficient of $x^n$ in $1 + \frac{a_e(x)^2}{2!} + \frac{a(x)^4}{4!} + \cdots$. 

Generating Functions
Now

\[
\sum_{k=0}^{\infty} \frac{a_e(x)^{2k}}{k!} = \frac{e^{a_e(x)} + e^{-a_e(x)}}{2} = \frac{1}{2} \left( \sqrt{\frac{1 + x}{1 - x}} + \sqrt{\frac{1 - x}{1 + x}} \right) = \frac{1}{\sqrt{1 - x^2}}
\]

Thus

\[
s_n = n! [x^n] \frac{1}{\sqrt{1 - x^2}} = \binom{n}{n/2} \frac{n!}{2^n}
\]
Exponential Families

- $P$ is a set referred to a set of pictures.
- A card $C$ is a pair $S, p$, where $p \in P$ and $S$ is a set of labels. The weight of $C$ is $n = |S|$. If $S = [n]$ then $C$ is a standard card.
- A hand $H$ is a set of cards whose label sets form a partition of $[n]$ for some $n \geq 1$. The weight of $H$ is $n$.
- $C' = (S', p)$ is a re-labelling of the card $C = (S, p)$ if $|S'| = |S|$.
- A deck $\mathcal{D}$ is a finite set of standard cards of common weight $n$, all of whose pictures are distinct.
- An exponential family $\mathcal{F}$ is a collection $\mathcal{D}_n, n \geq 1$, where the weight of $\mathcal{D}_n$ is $n$. 

Generating Functions
Given $\mathcal{F}$ let $h(n, k)$ denote the number of hands of weight $n$ consisting of $k$ cards, such that each card is a re-labelling of some card in some deck of $\mathcal{F}$.
(The same card can be used for re-labelling more than once.)

Next let the hand enumerator $H(x, y)$ be defined by

$$H(x, y) = \sum_{n \geq 1} \sum_{k \geq 0} h(n, k) \frac{x^n}{n!} y^k, \quad (h(n, 0) = 1_{n=0}).$$

Let $d_n = |D_n|$ and $D(x) = \sum_{n=1}^{\infty} \frac{d_n}{n!} x^n$.

**Theorem**

$$H(x, y) = e^{yD(x)}. \quad (5)$$
Example 1: Let $P = \{\text{directed cycles of all lengths}\}$.

A card is a directed cycle with labelled vertices.

A hand is a set of directed cycles of total length $n$ whose vertex labels partition $[n]$ i.e. it corresponds to a permutation of $[n]$.

$d_n = (n - 1)!$ and so

$$D(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = \log \left( \frac{1}{1 - x} \right)$$

and

$$H(x, y) = \exp \left\{ y \log \left( \frac{1}{1 - x} \right) \right\} = \frac{1}{(1 - x)^y}.$$
Let \( \begin{bmatrix} n \end{bmatrix} \) denote the number of permutations of \([n]\) with exactly \( k \) cycles. Then

\[
\sum_{k=1}^{n} \begin{bmatrix} n \end{bmatrix} y^k = \binom{x^n}{n!} \frac{1}{(1-x)^y} = n! \binom{y + n - 1}{n} = y(y+1) \cdots (y+n-1).
\]

The values \( \begin{bmatrix} n \end{bmatrix} \) are referred to as the Stirling numbers of the first kind.
Example 2: Let $P = \{[n], n \geq 1\}$.
A card is a non-empty set of positive integers.
A hand of $k$ cards is a partition of $[n]$ into $k$ non-empty subsets.
$d_n = 1$ for $n \geq 1$ and so

$$D(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1$$

and

$$\mathcal{H}(x, y) = e^{y(e^x-1)}.$$ 

So, if $\left\{ \binom{n}{k} \right\}$ is the number of partitions of $[n]$ into $k$ parts then

$$\left\{ \binom{n}{k} \right\} = \left[ \frac{x^n}{n!} \right] \frac{(e^x - 1)^k}{k!}.$$ 

The values $\left\{ \binom{n}{k} \right\}$ are referred to as the Stirling numbers of the second kind.
Proof of (5): Let $\mathcal{F}', \mathcal{F}''$ be two exponential families whose picture sets are disjoint. We merge them to form $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ by taking all $d'_n$ cards from the deck $\mathcal{D}'_n$ and adding them to the deck $\mathcal{D}''_n$ to make a deck of $d'_n + d''_n$ cards.

We claim that

$$\mathcal{H}(x, y) = \mathcal{H}'(x, y)\mathcal{H}''(x, y).$$

Indeed, a hand of $\mathcal{F}$ consists of $k'$ cards of total weight $n'$ together with $k'' = k - k'$ cards of total weight $n'' = n - n'$. The cards of $\mathcal{F}'$ will be labelled from an $n'$-subset $S$ of $[n]$. Thus,

$$h(n, k) = \sum_{n', k'} \binom{n}{n'} h'(n', k') h''(n - n', k - k').$$
But,

\[ \mathcal{H}'(x, y)\mathcal{H}''(x, y) = \left( \sum_{n', k'} h(n', k') \frac{x^{n'}}{n'!} y^{k'} \right) \left( \sum_{n'', k''} h(n'', k'') \frac{x^{n''}}{n''!} y^{k''} \right) = \sum_{n, k} \left( \frac{n!}{n'(n - n')!} h(n', k') h(n'', k'') \right) \frac{x^n}{n!} y^k. \]

This implies (6).
Now fix positive weights $r, d$ and consider an exponential family $\mathcal{F}_{r,d}$ that has $d$ cards in deck $D_{r}$ and no other non-empty decks. We claim that the hand enumerator of $\mathcal{F}_{r,d}$ is

$$
\mathcal{H}_{r,d}(x, y) = \exp \left\{ \frac{y dx^r}{r!} \right\}.
$$

(7)

We prove this by induction on $d$.

**Base Case $d = 1$:** A hand consists of $k \geq 0$ copies of the unique standard card that exists. If $n = kr$ then there are

$$
\binom{n!}{r!r! \ldots r!} = \frac{n!}{(r!)^k}
$$

choices for the labels of the cards. Then

$$
\mathcal{H}(kr, k) = \frac{1}{k!} \frac{n!}{(r!)^k}
$$

where we have divided by $k!$ because the cards in a hand are unordered. If $r$ does not divide $n$ then $\mathcal{H}(n, k) = 0$. 

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Thus,

\[
\mathcal{H}_{r,1}(x, y) = \sum_{k=0}^{\infty} \frac{1}{k! (r!)^k} \frac{n!}{n!} x^n y^k
\]

\[
= \exp \left\{ \frac{yx^r}{r!} \right\}
\]

**Inductive Step:** \( \mathcal{F}_{r,d} = \mathcal{F}_{r,1} \oplus \mathcal{F}_{r,d-1} \). So,

\[
\mathcal{H}_{r,d}(x, y) = \mathcal{H}_{r,1}(x, y) \mathcal{H}_{r,d-1}(x, y)
\]

\[
= \exp \left\{ \frac{yx^r}{r!} \right\} \exp \left\{ \frac{y(d-1)x^r}{r!} \right\}
\]

\[
= \exp \left\{ \frac{ydx^r}{r!} \right\},
\]

completing the induction.
Now consider a general deck $\mathcal{D}$ as the union of disjoint decks $\mathcal{D}_r, r \geq 1$. then,

$$H(x, y) = \prod_{r \geq 1} H_r(x, y) = \prod_{r \geq 1} \exp\left\{ \frac{ydx^r}{r!} \right\} = e^{y\mathcal{D}(x)}.$$