COMBINATORIAL GAMES
Start with $n$ chips. Players A, B alternately take 1, 2, 3 or 4 chips until there are none left. The winner is the person who takes the last chip:

Example

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>A</th>
<th>B</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 10$</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$n = 11$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

A wins

B wins

What is the optimal strategy for playing this game?
Chip placed at point \((m, n)\). Players can move chip to \((m', n)\) or \((m, n')\) where \(0 \leq m' < m\) and \(0 \leq n' < n\). The player who makes the last move and puts the chip onto \((0, 0)\) wins.

What is the optimal strategy for this game?

**Game 2a** Chip placed at point \((m, n)\). Players can move chip to \((m', n)\) or \((m, n')\) or to \((m - a, n - a)\) where \(0 \leq m' < m\) and \(0 \leq n' < n\) and \(0 \leq a \leq \min\{m, n\}\). The player who makes the last move and puts the chip onto \((0, 0)\) wins.

What is the optimal strategy for this game?
Game 3

$W$ is a set of words. A and B alternately remove words $w_1, w_2, \ldots$, from $W$. The rule is that the first letter of $w_{i+1}$ must be the same as the last letter of $w_i$. The player who makes the last legal move wins.

Example

$W = \{\text{England, France, Germany, Russia, Bulgaria, \ldots}\}$

What is the optimal strategy for this game?
Abstraction

Represent each position of the game by a vertex of a digraph $D = (X, A)$. 
$(x, y)$ is an arc of $D$ iff one can move from position $x$ to position $y$.

We assume that the digraph is finite and that it is acyclic i.e. there are no directed cycles.

The game starts with a token on vertex $x_0$ say, and players alternately move the token to $x_1, x_2, \ldots$, where $x_{i+1} \in N^+(x_i)$, the set of out-neighbours of $x_i$. The game ends when the token is on a sink i.e. a vertex of out-degree zero. The last player to move is the winner.
Example 1: \( V(D) = \{0, 1, \ldots, n\} \) and \((x, y) \in A\) iff \(x - y \in \{1, 2, 3, 4\}\).

Example 2: \( V(D) = \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\} \) and \((x, y) \in N^+((x', y'))\) iff \(x = x'\) and \(y > y'\) or \(x > x'\) and \(y = y'\).

Example 2a: \( V(D) = \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\} \) and \((x, y) \in N^+((x', y'))\) iff \(x = x'\) and \(y > y'\) or \(x > x'\) and \(y = y'\) or \(x - x' = y - y' > 0\).

Example 3: \( V(D) = \{(W', w) : W' \subseteq W \setminus \{w\}\}\). \(w\) is the last word used and \(W'\) is the remaining set of unused words. \((X', w') \in N^+((X, w))\) iff \(w' \in X\) and \(w'\) begins with the last letter of \(w\). Also, there is an arc from \((W, \cdot)\) to \((W \setminus \{w\}, w)\) for all \(w\), corresponding to the games start.
Abstraction

We will first argue that such a game must eventually end.

A **topological numbering** of digraph $D = (X, A)$ is a map $f : X \to [n]$, $n = |X|$ which satisfies $(x, y) \in A$ implies $f(x) < f(y)$.

**Theorem**

A finite digraph $D = (X, A)$ is acyclic iff it admits at least one topological numbering.

**Proof** Suppose first that $D$ has a topological numbering. We show that it is acyclic.

Suppose that $C = (x_1, x_2, \ldots, x_k, x_1)$ is a directed cycle. Then $f(x_1) < f(x_2) < \cdots < f(x_k) < f(x_1)$, contradiction.
Suppose now that $D$ is acyclic. We first argue that $D$ has at least one sink.

Thus let $P = (x_1, x_2, \ldots, x_k)$ be a longest simple path in $D$. We claim that $x_k$ is a sink.

If $D$ contains an arc $(x_k, y)$ then either $y = x_i, 1 \leq i \leq k - 1$ and this means that $D$ contains the cycle $(x_i, x_{i+1}, \ldots, x_k, x_i)$, contradiction or $y \notin \{x_1, x_2, \ldots, x_k\}$ and then $(P, y)$ is a longer simple path than $P$, contradiction.
We can now prove by induction on $n$ that there is at least one topological numbering.

If $n = 1$ and $X = \{x\}$ then $f(x) = 1$ defines a topological numbering.

Now assume that $n > 1$. Let $z$ be a sink of $D$ and define $f(z) = n$. The digraph $D' = D - z$ is acyclic and by the induction hypothesis it admits a topological numbering, $f : X \setminus \{z\} \to [n - 1]$.

The function we have defined on $X$ is a topological numbering. If $(x, y) \in A$ then either $x, y \neq z$ and then $f(x) < f(y)$ by our assumption on $f$, or $y = z$ and then $f(x) < n = f(z)$ ($x \neq z$ because $z$ is a sink).
The fact that $D$ has a topological numbering implies that the game must end. Each move increases the $f$ value of the current position by at least one and so after at most $n$ moves a sink must be reached.

The positions of a game are partitioned into 2 sets:

- $P$-positions: The next player cannot win. The previous player can win regardless of the current player’s strategy.
- $N$-positions: The next player has a strategy for winning the game.

Thus an $N$-position is a winning position for the next player and a $P$-position is a losing position for the next player.

The main problem is to determine $N$ and $P$ and what the strategy is for winning from an $N$-position.
Abstraction

Let the vertices of $D$ be $x_1, x_2, \ldots, x_n$, in topological order.

Labelling procedure

1. $i \leftarrow n$, Label $x_n$ with $P$. $N \leftarrow \emptyset$, $P \leftarrow \emptyset$.
2. $i \leftarrow i - 1$. If $i = 0$ STOP.
3. Label $x_i$ with $N$, if $N^+(x_i) \cap P \neq \emptyset$.
4. Label $x_i$ with $P$, if $N^+(x_i) \subseteq N$.
5. goto 2.

The partition $N, P$ satisfies

$$x \in N \text{ iff } N^+(x) \cap P \neq \emptyset$$

To play from $x \in N$, move to $y \in N^+(x) \cap P$. 

Combinatorial Games
Abstraction

In Game 1, \( P = \{5k : k \geq 0\} \).

In Game 2, \( P = \{(x, x) : x \geq 0\} \).

Lemma

The partition into \( N, P \) satisfying \( x \in N \) iff \( N^+(x) \cap P \neq \emptyset \) is unique.

Proof

If there were two partitions \( N_i, P_i, i = 1, 2 \), let \( x_i \) be the vertex of highest topological number which is not in \( (N_1 \cap N_2) \cup (P_1 \cap P_2) \). Suppose that \( x_i \in N_1 \setminus N_2 \).

But then \( x_i \in N_1 \) implies \( N^+(x_i) \cap P_1 \cap \{x_{i+1}, \ldots, x_n\} \neq \emptyset \) and \( x_i \in P_2 \) implies \( N^+(x_i) \cap P_2 \cap \{x_{i+1}, \ldots, x_n\} = \emptyset \).

But \( P_1 \cap \{x_{i+1}, \ldots, x_n\} = P_2 \cap \{x_{i+1}, \ldots, x_n\} \).
Suppose that we have \( p \) games \( G_1, G_2, \ldots, G_p \) with digraphs \( D_i = (X_i, A_i), i = 1, 2, \ldots, p \). The sum \( G_1 \oplus G_2 \oplus \cdots \oplus G_p \) of these games is played as follows. A position is a vector \((x_1, x_2, \ldots, x_p) \in X = X_1 \times X_2 \times \cdots \times X_p\). To make a move, a player chooses \( i \) such that \( x_i \) is not a sink of \( D_i \) and then replaces \( x_i \) by \( y \in N_i^+(x_i) \). The game ends when each \( x_i \) is a sink of \( D_i \) for \( i = 1, 2, \ldots, n \).

Knowing the partitions \( N_i, P_i \) for game \( i = 1, 2, \ldots, p \) does not seem to be enough to determine how to play the sum of the games.

We need more information. This will be provided by the Sprague-Grundy Numbering.
Example

Nim In a one pile game, we start with \( a \geq 0 \) chips and while there is a positive number \( x \) of chips, a move consists of deleting \( y \leq x \) chips. In this game the \( N \)-positions are the positive integers and the unique \( P \)-position is 0.

In general, Nim consists of the sum of \( n \) single pile games starting with \( a_1, a_2, \ldots, a_n > 0 \). A move consists of deleting some chips from a non-empty pile.

Example 2 is Nim with 2 piles.
Sprague-Grundy (SG) Numbering

For $S \subseteq \{0, 1, 2, \ldots, \}$ let

$$mex(S) = \min \{ x \geq 0 : x \notin S \}.$$

Now given an acyclic digraph $D = X, A$ with topological ordering $x_1, x_2, \ldots, x_n$ define $g$ iteratively by

1. $i \leftarrow n, g(x_n) = 0.$
2. $i \leftarrow i - 1.$ If $i = 0$ STOP.
3. $g(x_i) = mex(\{g(x) : x \in N^+(x_i)\}).$
4. goto 2.
Lemma

\[ x \in P \iff g(x) = 0. \]

Proof

Because

\[ x \in N \iff N^+(x) \cap P \neq \emptyset \]

all we have to show is that

\[ g(x) > 0 \iff \exists y \in N^+(y) \text{ such that } g(y) = 0. \]

But this is immediate from \( g(x) = \text{mex}(\{ g(y) : y \in N^+(x) \}) \) \( \square \)
Sums of games

Another one pile subtraction game.

- A player can remove any even number of chips, but not the whole pile.
- A player can remove the whole pile if it is odd.

The terminal positions are 0 or 2.

**Lemma**

\[ g(0) = 0, \ g(2k) = k - 1 \text{ and } g(2k - 1) = k \text{ for } k \geq 1. \]
Proof 0, 2 are terminal positions and so \( g(0) = g(2) = 0 \). \( g(1) = 1 \) because the only position one can move to from 1 is 0. We prove the remainder by induction on \( k \).

Assume that \( k > 1 \).

\[
\begin{align*}
g(2k) & = \text{mex}\{g(2k - 2), g(2k - 4), \ldots, g(2)\} \\
& = \text{mex}\{k - 2, k - 3, \ldots, 0\} \\
& = k - 1.
\end{align*}
\]

\[
\begin{align*}
g(2k - 1) & = \text{mex}\{g(2k - 3), g(2k - 5), \ldots, g(1), g(0)\} \\
& = \text{mex}\{k - 1, k - 2, \ldots, 0\} \\
& = k.
\end{align*}
\]
Sums of games

We now show how to compute the \( SG \) numbering for a sum of games.

For binary integers \( a = a_m a_{m-1} \cdots a_1 a_0 \) and \( b = b_m b_{m-1} \cdots b_1 b_0 \) we define \( a \oplus b = c_m c_{m-1} \cdots c_1 c_0 \) by

\[
c_i = \begin{cases} 
1 & \text{if } a_i \neq b_i \\
0 & \text{if } a_i = b_i
\end{cases}
\]

for \( i = 1, 2, \ldots, m \).

So \( 11 \oplus 5 = 14 \).
Sums of games

Theorem

If $g_i$ is the SG function for game $G_i$, $i = 1, 2, \ldots, p$ then the SG function $g$ for the sum of the games $G = G_1 \oplus G_2 \oplus \cdots \oplus G_p$ is defined by

$$g(x) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_p(x_p)$$

where $x = (x_1, x_2, \ldots, x_p)$.

For example if in a game of Nim, the pile sizes are $x_1, x_2, \ldots, x_p$ then the SG value of the position is

$$x_1 \oplus x_2 \oplus \cdots \oplus x_p$$
Proof  It is enough to show this for $p = 2$ and then use induction on $p$.

Write $G = H \oplus G_p$ where $H = G_1 \oplus G_2 \oplus \cdots \oplus G_{p-1}$. Let $h$ be the $SG$ numbering for $H$. Then, if $y = (x_1, x_2, \ldots, x_{p-1})$,

$$g(x) = h(y) \oplus g_p(x_p) \quad \text{assuming theorem for } p = 2$$

$$= g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_{p-1}(x_{p-1}) \oplus g_p(x_p)$$

by induction.

It is enough now to show, for $p = 2$, that

A1  If $x \in X$ and $g(x) = b > a$ then there exists $x' \in N^+(x)$ such that $g(x') = a$.

A2  If $x \in X$ and $g(x) = b$ and $x' \in N^+(x)$ then $g(x') \neq g(x)$.

A3  If $x \in X$ and $g(x) = 0$ and $x' \in N^+(x)$ then $g(x') \neq 0$
A1. Write $d = a \oplus b$. Then

$$a = d \oplus b = d \oplus g_1(x_1) \oplus g_2(x_2). \quad (1)$$

Now suppose that we can show that either

(i) $d \oplus g_1(x_1) < g_1(x_1)$ or (ii) $d \oplus g_2(x_2) < g_2(x_2)$ or both. (2)

Assume that (i) holds.

Then since $g_1(x_1) = \text{mex}(N_1^+(x_1))$ there must exist $x'_1 \in N_1^+(x_1)$ such that $g_1(x'_1) = d \oplus g_1(x_1)$.

Then from (1) we have

$$a = g_1(x'_1) \oplus g_2(x_2) = g(x'_1, x_2).$$

Furthermore, $(x'_1, x_2) \in N^+(x)$ and so we will have verified A1.
Let us verify (2).

Suppose that $2^{k-1} \leq d < 2^k$.

Then $d$ has a 1 in position $k$ and no higher.

Since $d_k = a_k \oplus b_k$ and $a < b$ we must have $a_k = 0$ and $b_k = 1$.

So either (i) $g_1(x_1)$ has a 1 in position $k$ or (ii) $g_2(x_2)$ has a 1 in position $k$. Assume (i).

But then $d \oplus g_1(x_1) < g_1(x_1)$ since $d$ “destroys” the $k$th bit of $g_1(x_1)$ and does not change any higher bit.
A2. Suppose without loss of generality that $g(x'_1, x_2) = g(x_1, x_2)$ where $x'_1 \in N^+(x_1)$.

Then $g_1(x'_1) \oplus g_2(x_2) = g_1(x_1) \oplus g_2(x_2)$ implies that $g_1(x'_1) = g_1(x_1)$, contradiction. □

A3. Suppose that $g_1(x_1) \oplus g_2(x_2) = 0$ and $g_1(x'_1) \oplus g_2(x_2) = 0$ where $x'_1 \in N^+(x_1)$.

Then $g_1(x_1) = g_1(x'_1)$, contradicting $g_1(x_1) = \text{mex}\{g_1(x) : x \in N^+(x_1)\}$. 
If we apply this theorem to the game of Nim then if the position \( x \) consists of piles of \( x_i \) chips for \( i = 1, 2, \ldots, p \) then 
\[
g(x) = x_1 \oplus x_2 \oplus \cdots \oplus x_p.
\]

In our first example, \( g(x) = x \mod 5 \) and so for the sum of \( p \) such games we have 
\[
g(x_1, x_2, \ldots, x_p) = (x_1 \mod 5) \oplus (x_2 \mod 5) \oplus \cdots \oplus (x_p \mod 5).
\]
A more complicated one pile game

Start with $n$ chips. First player can remove up to $n - 1$ chips.

In general, if the previous player took $x$ chips, then the next player can take $y \leq x$ chips.

Thus a games position can be represented by $(n, x)$ where $n$ is the current size of the pile and $x$ is the maximum number of chips that can be removed in this round.

**Theorem**

Suppose that the position is $(n, x)$ where $n = m2^k$ and $m$ is odd. Then,

(a) This is an *N*-position if $x \geq 2^k$.

(b) This is a *P*-position if $m = 1$ and $x < n$. 
A more complicated one pile game

Proof  For a non-negative integer \( n = m2^k \), let \( \text{ones}(n) \) denote the number of ones in the binary expansion of \( n \) and let \( k = \rho(n) \) determine the position of the right-most one in this expansion.

We claim that the following strategy is a win for the player in a position described in (a):

Remove \( y = 2^k \) chips.

Suppose this player is A.

If \( m = 1 \) then \( x \geq n \) and A wins.
A more complicated one pile game

Otherwise, after such a move the position is \((n', y)\) where \(\rho(n') > \rho(n)\).

Note first that \(\text{ones}(n') = \text{ones}(n) - 1 > 0\) and \(\rho(n') > k\). B cannot remove more than \(2^k\) chips and so B cannot win at this point.

If B moves the position to \((n'', x'')\) then \(\text{ones}(n'') > \text{ones}(n')\) and furthermore, \(x'' \geq 2^\rho(n'')\), since \(x''\) must have a 1 in position \(\rho(n'')\). (\(\rho(n'')\) is the least significant bit of \(x''\).)

Thus, by induction, A is in an \(N\)-position and wins the game.

To prove (b), note that after the first move, the position satisfies the conditions of (a). \(\square\).
Geography

Start with a chip sitting on a vertex $v$ of a graph or digraph $G$. A move consists of moving the chip to a neighbouring vertex.

In edge geography, moving the chip from $x$ to $y$ deletes the edge $(x, y)$. In vertex geography, moving the chip from $x$ to $y$ deletes the vertex $x$.

The problem is given a position $(G, v)$, to determine whether this is a $P$ or $N$ position.

**Complexity** Both edge and vertex geography are Pspace-hard on digraphs. Edge geography is Pspace-hard on an undirected graph. Only vertex geography on a graph is polynomial time solvable.
We need some simple results from the theory of matchings on graphs.
A *matching* $M$ of a graph $G = (V, E)$ is a set of edges, no two of which are incident to a common vertex.
An $M$-alternating path joining 2 $M$-unsaturated vertices is called an $M$-augmenting path.
**Theorem**

*M* is a *maximum* matching iff *M* admits no *M*-augmenting paths.

**Proof**

Suppose *M* has an augmenting path

\[ P = (a_0, b_1, a_1, \ldots, a_k, b_{k+1}) \]

where

\[ e_i = (a_{i-1}, b_i) \notin M, \; 1 \leq i \leq k + 1 \]

and

\[ f_i = (b_i, a_i) \in M, \; 1 \leq i \leq k. \]

Let \( M' = M - \{f_1, f_2, \ldots, f_k\} + \{e_1, e_2, \ldots, e_{k+1}\} \).
Undirected Vertex Geography

- $|M'| = |M| + 1$.
- $M'$ is a matching

For $x \in V$ let $d_M(x)$ denote the degree of $x$ in matching $M$, So $d_M(x)$ is 0 or 1.

$$d_{M'}(x) = \begin{cases} 
    d_M(x) & x \notin \{a_0, b_1, \ldots, b_{k+1}\} \\
    d_M(x) & x \in \{b_1, \ldots, a_k\} \\
    d_M(x) + 1 & x \in \{a_0, b_{k+1}\}
\end{cases}$$

So if $M$ has an augmenting path it is not maximum.
Suppose $M$ is not a maximum matching and $|M'| > |M|$. Consider $H = G[M \vartriangle M']$ where $M \vartriangle M' = (M \setminus M') \cup (M' \setminus M)$ is the set of edges in exactly one of $M, M'$. Maximum degree of $H$ is $2 - \leq 1$ edge from $M$ or $M'$. So $H$ is a collection of vertex disjoint alternating paths and cycles.

$|M'| > |M|$ implies that there is at least one path of type (d). Such a path is $M$-augmenting.
Theorem

\((G, v)\) is an N-position in UVG iff every maximum matching of \(G\) covers \(v\).

Proof  (i) Suppose that \(M\) is a maximum matching of \(G\) which covers \(v\). Player 1’s strategy is now: Move along the \(M\)-edge that contains the current vertex.

If Player 1 were to lose, then there would exist a sequence of edges \(e_1, f_1, \ldots, e_k, f_k\) such that \(v \in e_1, e_1, e_2, \ldots, e_k \in M, f_1, f_2, \ldots, f_k \not\in M\) and \(f_k = (x, y)\) where \(y\) is the current vertex for Player 1 and \(y\) is not covered by \(M\).

But then if \(A = \{e_1, e_2, \ldots, e_k\}\) and \(B = \{f_1, f_2, \ldots, f_k\}\) then \((M \setminus A) \cup B\) is a maximum matching (same size as \(M\)) which does not cover \(v\), contradiction.
(ii) Suppose now that there is some maximum matching $M$ which does not cover $v$. If $(v, w)$ is Player 1’s move, then $w$ must be covered by $M$, else $M$ is not a maximum matching.

Player 2’s strategy is now: Move along the $M$-edge that contains the current vertex. If Player 2 were to lose then there exists $e_1 = (v, w), f_1, \ldots, e_k, f_k, e_{k+1} = (x, y)$ where $y$ is the current vertex for Player 2 and $y$ is not covered by $M$.

But then we have defined an augmenting path from $v$ to $y$ and so $M$ is not a maximum matching, contradiction.
Note that we can determine whether or not $v$ is covered by all maximum matchings as follows: Find the size $\sigma$ of the maximum matching $G$.

This can be done in $O(n^3)$ time on an $n$-vertex graph. Find the size $\sigma'$ of a maximum matching in $G - v$. Then $v$ is covered by all maximum matchings of $G$ iff $\sigma \neq \sigma'$. 
An even kernel of $G$ is a non-empty set $S \subseteq V$ such that (i) $S$ is an independent set and (ii) $v \notin S$ implies that $\deg_S(v)$ is even, (possibly zero). ($\deg_S(v)$ is the number of neighbours of $v$ in $S$.)

Lemma

If $S$ is an even kernel and $v \in S$ then $(G, v)$ is a P-position in UEG.

Proof Any move at a vertex in $S$ takes the chip outside $S$ and then Player 2 can immediately put the chip back in $S$. After a move from $x \in S$ to $y \notin S$, $\deg_S(y)$ will become odd and so there is an edge back to $S$. making this move, makes $\deg_S(y)$ even again. Eventually, there will be no $S : \bar{S}$ edges and Player 1 will be stuck in $S$. □
We now discuss Bipartite UEG i.e. we assume that $G$ is bipartite, $G$ has bipartition consisting of a copy of $[m]$ and a disjoint copy of $[n]$ and edges set $E$. Now consider the $m \times n$ 0-1 matrix $A$ with $A(i,j) = 1$ iff $(i,j) \in E$.

We can play our game on this matrix: We are either positioned at row $i$ or we are positioned at column $j$. If say, we are positioned at row $i$, then we choose a $j$ such that $A(i,j) = 1$ and (i) make $A(i,j) = 0$ and (ii) move the position to column $j$. An analogous move is taken when we positioned at column $j$.

**Lemma**

*Suppose the current position is row $i$. This is a P-position iff row $i$ is in the span of the remaining rows (is the sum (mod 2) of a subset of the other rows) or row $i$ is a zero row. A similar statement can be made if the position is column $j$.***
**Proof**  If row $i$ is a zero row then vertex $i$ is isolated and this is clearly a P-position. Otherwise, assume the position is row 1 and there exists $I \subseteq [m]$ such that $1 \in I$ and

$$r_1 = \sum_{i \in I \setminus \{1\}} r_i \pmod{2} \text{ or } \sum_{i \in I} r_i = 0 \pmod{2} \quad (3)$$

where $r_i$ denotes row $i$.

$I$ is an even kernel: If $x \not\in I$ then either (i) $x$ corresponds to a row and there are no $x, I$ edges or (ii) $x$ corresponds to a column and then $\sum_{i \in I} A(i, x) = 0 \pmod{2}$ from (3) and then $x$ has an even number of neighbours in $I$. 
Now suppose that (3) does not hold for any \( l \). We show that there exists a \( l \) such that \( A(1, l) = 1 \) and putting \( A(1, l) = 0 \) makes column \( l \) dependent on the remaining columns. Then we will be in a P-position, by the first part.

Let \( e_1 \) be the \( m \)-vector with a 1 in row 1 and a 0 everywhere else. Let \( A^* \) be obtained by adding \( e_1 \) to \( A \) as an \((n + 1)\)th column. Now the row-rank of \( A^* \) is the same as the row-rank of \( A \) (here we are doing all arithmetic modulo 2). Suppose not, then if \( r_i^* \) is the \( i \)th row of \( A^* \) then there exists a set \( J \) such that

\[
\sum_{i \in J} r_i = 0 \pmod{2} \neq \sum_{i \in J} r_i^* \pmod{2}.
\]

Now \( 1 \notin J \) because \( r_1 \) is independent of the remaining rows of \( A \), but then \( \sum_{i \in J} r_i = 0 \pmod{2} \) implies \( \sum_{i \in J} r_i^* = 0 \pmod{2} \) since the last column has all zeros, except in row 1.
Thus rank $A^* = \text{rank } A$ and so there exists $K \subseteq [n]$ such that

$$e_1 = \sum_{k \in K} c_k \pmod{2} \text{ or } e_1 + \sum_{k \in K} c_k = 0 \pmod{2} \quad (4)$$

where $c_k$ denotes column $k$ of $A$.

Thus there exists $\ell \in K$ such that $A(1, \ell) = 1$. Now let $c'_j = c_j$ for $j \neq \ell$ and $c'_\ell$ be obtained from $c_\ell$ by putting $A(1, \ell) = 0$ i.e. $c'_\ell = c_\ell + e_1$. But then (4) implies that $\sum_{k \in K} c'_k = 0 \pmod{2}$ ($K = \{ k \}$ is a possibility here).
We consider the following multi-dimensional version of Tic Tac Toe (Noughts and Crosses to the English).

The *board* consists of $[n]^d$. A point on the board is therefore a vector $(x_1, x_2, \ldots, x_d)$ where $1 \leq x_i \leq n$ for $1 \leq i \leq d$.

A *line* is a set points $(x_{j}^{(1)}, x_{j}^{(2)}, \ldots, x_{j}^{(d)}), j = 1, 2, \ldots, n$ where each sequence $x^{(i)}$ is either (i) of the form $k, k, \ldots, k$ for some $k \in [n]$ or is (ii) $1, 2, \ldots, n$ or is (iii) $n, n - 1, \ldots, 1$. Finally, we cannot have Case (i) for all $i$.

Thus in the (familiar) $3 \times 3$ case, the top row is defined by $x^{(1)} = 1, 1, 1$ and $x^{(2)} = 1, 2, 3$ and the diagonal from the bottom left to the top right is defined by $x^{(1)} = 3, 2, 1$ and $x^{(2)} = 1, 2, 3$. 
Lemma

The number of winning lines in the \((n, d)\) game is \(\frac{(n+2)^d - n^d}{2}\).

Proof  In the definition of a line there are \(n\) choices for \(k\) in (i) and then (ii), (iii) make it up to \(n + 2\). There are \(d\) independent choices for each \(i\) making \((n + 2)^d\).

Now delete \(n^d\) choices where only Case (i) is used. Then divide by 2 because replacing (ii) by (iii) and vice-versa whenever Case (i) does not hold produces the same set of points (traversing the line in the other direction). □
The game is played by 2 players. The Red player (X player) goes first and colours a point red. Then the Blue player (0 player) colours a different point blue and so on.

A player wins if there is a line, all of whose points are that players colour. If neither player wins then the game is a draw. The second player does not have a winning strategy:

**Lemma**

*Player 1 can always get at least a draw.*
**Proof** We prove this by considering *strategy stealing*.

Suppose that Player 2 did have a winning strategy. Then Player 1 can make an arbitrary first move $x_1$. Player 2 will then move with $y_1$. Player 1 will now win playing the winning strategy for Player 2 against a first move of $y_1$.

This can be carried out until the strategy calls for move $x_1$ (if at all). But then Player 1 can make an arbitrary move and continue, since $x_1$ has already been made. □
The above array gives a strategy for Player 2 the $5 \times 5$ game ($d = 2, n = 5$).

For each of the 12 lines there is an associated pair of positions. If Player 1 chooses a position with a number $i$, then Player 2 responds by choosing the other cell with the number $i$.

This ensures that Player 1 cannot take line $i$. If Player 1 chooses the * then Player 2 can choose any cell with an unused number.
Pairing Strategy

So, later in the game if Player 1 chooses a cell with \( j \) and Player 2 already has the other \( j \), then Player 2 can choose an arbitrary cell.

Player 2’s strategy is to ensure that after all cells have been chosen, he/she will have chosen one of the numbered cells associated with each line. This prevents Player 1 from taking a whole line. This is called a *pairing* strategy.
We now generalise the game to the following: We have a family $\mathcal{F} = A_1, A_2, \ldots, A_N \subseteq A$. A move consists of one player, taking an uncoloured member of $A$ and giving it his colour.

A player wins if one of the sets $A_i$ is completely coloured with his colour.

A pairing strategy is a collection of distinct elements $X = \{x_1, x_2, \ldots, x_{2N-1}, x_{2N}\}$ such that $x_{2i-1}, x_{2i} \in A_i$ for $i \geq 1$.

This is called a *draw forcing pairing*. Player 2 responds to Player 1’s choice of $x_{2i+\delta}$, $\delta = 0, 1$ by choosing $x_{2i+3-\delta}$. If Player 1 does not choose from $X$, then Player 2 can choose any uncoloured element of $X$. 
In this way, Player 2 avoids defeat, because at the end of the game Player 2 will have coloured at least one of each of the pairs $x_{2i-1}, x_{2i}$ and so Player 1 cannot have completely coloured $A_i$ for $i = 1, 2, \ldots, N$. 
Theorem

If

\[ \left| \bigcup_{A \in G} A \right| \geq 2|G| \quad \forall G \subseteq F \]  

(5)

then there is a draw forcing pairing.

Proof  
We define a bipartite graph $\Gamma$. $A$ will be one side of the bipartition and $B = \{b_1, b_2, \ldots, b_{2N}\}$. Here $b_{2i - 1}$ and $b_{2i}$ both represent $A_i$ in the sense that if $a \in A_i$ then there is an edge $(a, b_{2i - 1})$ and an edge $(a, b_{2i})$.

A draw forcing pairing corresponds to a complete matching of $B$ into $A$ and the condition (5) implies that Hall’s condition is satisfied.
**Corollary**

If $|A_i| \geq n$ for $i = 1, 2, \ldots, n$ and every $x \in A$ is contained in at most $n/2$ sets of $\mathcal{F}$ then there is a draw forcing pairing.

**Proof** The degree of $a \in A$ is at most $2(n/2)$ in $\Gamma$ and the degree of each $b \in B$ is at least $n$. This implies (via Hall’s condition) that there is a complete matching of $B$ into $A$. □
Consider Tic tac Toe when $d = 2$. If $n$ is even then every array element is in at most 3 lines (one row, one column and at most one diagonal) and if $n$ is odd then every array element is in at most 4 lines (one row, one column and at most two diagonals).

Thus there is a draw forcing pairing if $n \geq 6$, $n$ even and if $n \geq 9$, $n$ odd. (The cases $n = 4, 7$ have been settled as draws. $n = 7$ required the use of a computer to examine all possible strategies.)
In general we have

**Lemma**

If \( n \geq 3^d - 1 \) and \( n \) is odd or if \( n \geq 2^d - 1 \) and \( n \) is even, then there is a draw forcing pairing of \((n, d)\) Tic tac Toe.

**Proof**

We only have to estimate the number of lines through a fixed point \( c = (c_1, c_2, \ldots, c_d) \).

If \( n \) is odd then to choose a line \( L \) through \( c \) we specify, for each index \( i \) whether \( L \) is (i) constant on \( i \), (ii) increasing on \( i \) or (iii) decreasing on \( i \).

This gives \( 3^d \) choices. Subtract 1 to avoid the all constant case and divide by 2 because each line gets counted twice this way.
When $n$ is even, we observe that once we have chosen in which positions $L$ is constant, $L$ is determined.

Suppose $c_1 = x$ and 1 is not a fixed position. Then every other non-fixed position is $x$ or $n - x + 1$. Assuming w.l.o.g. that $x \leq n/2$ we see that $x < n - x = 1$ and the positions with $x$ increase together at the same time as the positions with $n - x + 1$ decrease together.

Thus the number of lines through $c$ in this case is bounded by $\sum_{i=0}^{d-1} \binom{d}{i} = 2^d - 1$. □
We now prove a theorem of Erdős and Selfridge.

**Theorem**

If \(|A_i| \geq n\) for \(i \in [N]\) and \(N < 2^{n-1}\), then Player 2 can get a draw in the game defined by \(\mathcal{F}\).

**Proof**

At any point in the game, let \(C_j\) denote the set of elements in \(A\) which have been coloured with Player \(j\)'s colour, \(j = 1, 2\) and \(U = A \setminus C_1 \cup C_2\). Let

\[
\Phi = \sum_{i: A_i \cap C_2 = \emptyset} 2^{-|A_i \cap U|}.
\]

Suppose that the players choices are \(x_1, y_1, x_2, y_2, \ldots\). Then we observe that immediately after Player 1’s first move, \(\Phi < N2^{-(n-1)} < 1\).
We will show that Player 2 can keep $\Phi < 1$ throughout. Then at the end, when $U = \emptyset$, $\Phi = \sum_{i: A_i \cap C_2 = \emptyset} 1 < 1$ implies that $A_i \cap C_2 \neq \emptyset$ for all $i \in [N]$.

So, now let $\Phi_j$ be the value of $\Phi$ after the choice of $x_1, y_1, \ldots, x_j$. then if $U, C_1, C_2$ are defined at precisely this time,

$$
\Phi_{j+1} - \Phi_j = - \sum_{i: A_i \cap C_2 = \emptyset \atop y_j \in A_i} 2^{-|A_i \cap U|} + \sum_{i: A_i \cap C_2 = \emptyset \atop y_j \notin A_i, x_{j+1} \in A_i} 2^{-|A_i \cap U|}
$$

$$
\leq - \sum_{i: A_i \cap C_2 = \emptyset \atop y_j \in A_i} 2^{-|A_i \cap U|} + \sum_{i: A_i \cap C_2 = \emptyset \atop x_{j+1} \in A_i} 2^{-|A_i \cap U|}
$$
We deduce that $\Phi_{j+1} - \Phi_j \leq 0$ if Player 2 chooses $y_j$ to maximise $\sum_{i:A_i \cap C_2 = \emptyset} 2^{-|A_i \cap U|}$ over $y$.

In this way, Player 2 keeps $\Phi < 1$ and obtains a draw. $\Box$

In the case of $(n, d)$ Tic Tac Toe, we see that Player 2 can force a draw if

$$\frac{(n + 2)^d - n^d}{2} < 2^{n-1}$$

which is implied, for $n$ large, by

$$n \geq (1 + \epsilon)d \log_2 d$$

where $\epsilon > 0$ is a small positive constant.