BASIC COUNTING
Let $\phi(m, n)$ be the number of mappings from $[n]$ to $[m]$.

**Theorem**

$$
\phi(m, n) = m^n
$$

**Proof**  
By induction on $n$.

$$
\phi(m, 0) = 1 = m^0.
$$

$$
\begin{align*}
\phi(m, n + 1) &= m \phi(m, n) \\
&= m \times m^n \\
&= m^{n+1}.
\end{align*}
$$

$\phi(m, n)$ is also the number of sequences $x_1 x_2 \cdots x_n$ where $x_i \in [m]$, $i = 1, 2, \ldots, n$. 

Basic Counting
Let $\psi(n)$ be the number of subsets of $[n]$.

**Theorem**

$$\psi(n) = 2^n.$$  

**Proof**  
(1) By induction on $n$.  
$\psi(0) = 1 = 2^0$. 

$$\psi(n+1) = \#\{\text{sets containing } n + 1\} + \#\{\text{sets not containing } n + 1\}$$  
$$= \psi(n) + \psi(n)$$  
$$= 2^n + 2^n$$  
$$= 2^{n+1}.$$
There is a general principle that if there is a 1-1 correspondence between two finite sets $A, B$ then $|A| = |B|$. Here is a use of this principle.

**Proof (2).**
For $A \subseteq [n]$ define the map $f_A : [n] \rightarrow \{0, 1\}$ by

$$f_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$ 

$f_A$ is the characteristic function of $A$.

Distinct $A$’s give rise to distinct $f_A$’s and vice-versa.

Thus $\psi(n)$ is the number of choices for $f_A$, which is $2^n$ by Theorem 1. \qed
Let $\psi_{\text{odd}}(n)$ be the number of odd subsets of $[n]$ and let $\psi_{\text{even}}(n)$ be the number of even subsets.

**Theorem**

\[
\psi_{\text{odd}}(n) = \psi_{\text{even}}(n) = 2^{n-1}.
\]

**Proof** For $A \subseteq [n-1]$ define

\[
A' = \begin{cases} 
A & |A| \text{ is odd} \\
A \cup \{n\} & |A| \text{ is even}
\end{cases}
\]

The map $A \rightarrow A'$ defines a bijection between $[n-1]$ and the odd subsets of $[n]$. So $2^{n-1} = \psi(n-1) = \psi_{\text{odd}}(n)$. Furthermore,

\[
\psi_{\text{even}}(n) = \psi(n) - \psi_{\text{odd}}(n) = 2^n - 2^{n-1} = 2^{n-1}.
\]
Let $\phi_{1-1}(m, n)$ be the number of 1-1 mappings from $[n]$ to $[m]$.

**Theorem**

\[
\phi_{1-1}(m, n) = \prod_{i=0}^{n-1} (m - i). \tag{1}
\]

**Proof**  Denote the RHS of (1) by $\pi(m, n)$. If $m < n$ then $\phi_{1-1}(m, n) = \pi(m, n) = 0$. So assume that $m \geq n$. Now we use induction on $n$.

If $n = 0$ then we have $\phi_{1-1}(m, 0) = \pi(m, 0) = 1$.

In general, if $n < m$ then

\[
\phi_{1-1}(m, n + 1) = (m - n) \phi_{1-1}(m, n) \\
= (m - n) \pi(m, n) \\
= \pi(m, n + 1).
\]
\( \phi_{1-1}(m, n) \) also counts the number of length \( n \) ordered sequences distinct elements taken from a set of size \( m \).

\[ \phi_{1-1}(n, n) = n(n-1) \cdots 1 = n! \]

is the number of ordered sequences of \([n]\) i.e. the number of permutations of \([n]\).
Binomial Coefficients

\[
\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1) \cdots (n-k+1)}{k(k-1) \cdots 1}
\]

Let \( X \) be a finite set and let

\[
\binom{X}{k}
\]

denote the collection of \( k \)-subsets of \( X \).

**Theorem**

\[
\left| \binom{X}{k} \right| = \binom{|X|}{k}.
\]

**Proof**

Let \( n = |X|, \)

\[
k! \left| \binom{X}{k} \right| = \phi_{1-1}(n, k) = n(n-1) \cdots (n-k+1).
\]
Let $m, n$ be non-negative integers. Let $\mathbb{Z}_+^n$ denote the non-negative integers. Let

$$S(m, n) = \{(i_1, i_2, \ldots, i_n) \in \mathbb{Z}_+^n : i_1 + i_2 + \cdots + i_n = m\}.$$

**Theorem**

$$|S(m, n)| = \binom{m + n - 1}{n - 1}.$$

**Proof** imagine $m + n - 1$ points in a line. Choose positions $p_1 < p_2 < \cdots < p_{n-1}$ and color these points red. Let $p_0 = 0$, $p_n = m + 1$. The gap sizes between the red points

$$i_t = p_t - p_{t-1} - 1, \ t = 1, 2, \ldots, n$$

form a sequence in $S(m, n)$ and vice-versa. \qed
\(|S(m, n)|\) is also the number of ways of coloring \(m\) indistinguishable balls using \(n\) colors.

Suppose that we want to count the number of ways of coloring these balls so that each color appears at least once i.e. to compute \(|S(m, n)^*|\) where, if \(N = \{1, 2, \ldots, \}\)

\[
S(m, n)^* = \\
\{(i_1, i_2, \ldots, i_n) \in N^n : i_1 + i_2 + \cdots + i_n = m\} \\
= \{(i_1 - 1, i_2 - 1, \ldots, i_n - 1) \in Z_+^n : \\
(i_1 - 1) + (i_2 - 1) + \cdots + (i_n - 1) = m - n\}
\]

Thus,

\[
|S(m, n)^*| = \binom{m - n + n - 1}{n - 1} = \binom{m - 1}{n - 1}.
\]
How many ways (patterns) are there of placing $k$ 1’s and $n - k$ 0’s at the vertices of a polygon with $n$ vertices so that no two 1’s are adjacent?

Choose a vertex $v$ of the polygon in $n$ ways and then place a 1 there. For the remainder we must choose $a_1, \ldots, a_k \geq 1$ such that $a_1 + \cdots + a_k = n - k$ and then go round the cycle (clockwise) putting $a_1$ 0’s followed by a 1 and then $a_2$ 0’s followed by a 1 etc..
Each pattern $\pi$ arises $k$ times in this way. There are $k$ choices of $v$ that correspond to a 1 of the pattern. Having chosen $v$ there is a unique choice of $a_1, a_2, \ldots, a_k$ that will now give $\pi$.

There are $\binom{n-k-1}{k-1}$ ways of choosing the $a_i$ and so the answer to our question is

$$\frac{n}{k} \binom{n-k-1}{k-1}.$$
Theorem

**Symmetry**

\[
\binom{n}{r} = \binom{n}{n-r}
\]

**Proof**
Choosing \( r \) elements to include is equivalent to choosing \( n - r \) elements to exclude. \(\square\)
Theorem

Pascal’s Triangle

\[
\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}
\]

Proof

A \(k + 1\)-subset of \([n + 1]\) either
(i) includes \(n + 1\) —— \(\binom{n}{k}\) choices or
(ii) does not include \(n + 1\) —— \(\binom{n}{k+1}\) choices.
Pascal’s Triangle
The following array of binomial coefficients, constitutes the famous triangle:

```
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
...```
Theorem

\[
\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}. \quad (2)
\]

Proof

1: Induction on \( n \) for arbitrary \( k \).

Base case: \( n = k \); \( \binom{k}{k} = \binom{k+1}{k+1} \)

Inductive Step: assume true for \( n \geq k \).

\[
\sum_{m=k}^{n+1} \binom{m}{k} = \sum_{m=k}^{n} \binom{m}{k} + \binom{n+1}{k}
\]

\[
= \binom{n+1}{k+1} + \binom{n+1}{k} \quad \text{Induction}
\]

\[
= \binom{n+2}{k+1}. \quad \text{Pascal’s triangle}
\]
Proof 2: Combinatorial argument.
If $S$ denotes the set of $k + 1$-subsets of $[n + 1]$ and $S_m$ is the set of $k + 1$-subsets of $[n + 1]$ which have largest element $m + 1$ then

- $S_k, S_{k+1}, \ldots, S_n$ is a partition of $S$.
- $|S_k| + |S_{k+1}| + \cdots + |S_n| = |S|$.
- $|S_m| = \binom{m}{k}$.

□
**Theorem**

**Vandermonde’s Identity**

\[
\sum_{r=0}^{k} \binom{m}{r} \binom{n}{k-r} = \binom{m+n}{k}.
\]

**Proof**

Split \([m+n]\) into \(A = [m]\) and \(B = [m+n] \setminus [m]\). Let \(S\) denote the set of \(k\)-subsets of \([m+n]\) and let \(S_r = \{ X \in S : |X \cap A| = r \}\). Then

- \(S_0, S_1, \ldots, S_k\) is a partition of \(S\).
- \(|S_0| + |S_1| + \cdots + |S_k| = |S|\).
- \(|S_r| = \binom{m}{r} \binom{n}{k-r}\).
- \(|S| = \binom{m+n}{k}\). 

\(\square\)
**Theorem**

*Binomial Theorem*

\[(1 + x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.\]

**Proof**  
Coefficient \(x^r\) in \((1 + x)(1 + x) \cdots (1 + x)\): choose \(x\) from \(r\) brackets and 1 from the rest.  
□
Applications of Binomial Theorem

• $x = 1$:

\[
\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = (1 + 1)^n = 2^n.
\]

LHS counts the number of subsets of all sizes in $[n]$.

• $x = -1$:

\[
\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = (1 - 1)^n = 0,
\]

i.e.

\[
\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots
\]

and number of subsets of even cardinality = number of subsets of odd cardinality.
\[ \sum_{k=0}^{n} \binom{n}{k} k = n2^{n-1}. \]

Differentiate both sides of the Binomial Theorem w.r.t. \( x \).

\[ n(1 + x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} x^{k-1} \]

Now put \( x = 1 \).
Grid path problems

A monotone path is made up of segments 
\((x, y) \rightarrow (x + 1, y)\) or \((x, y) \rightarrow (x, y + 1)\).

\((a, b) \rightarrow (c, d))= \{\text{monotone paths from } (a, b) \text{ to } (c, d)\}.

We drop the \( (a, b) \rightarrow \) for paths starting at \( (0, 0) \).
We consider 3 questions: Assume $a, b \geq 0$.

1. How large is $\text{PATHS}(a, b)$?

2. Assume $a < b$. Let $\text{PATHS}_>(a, b)$ be the set of paths in $\text{PATHS}(a, b)$ which do not touch the line $x = y$ except at $(0, 0)$. How large is $\text{PATHS}_>(a, b)$?

3. Assume $a \leq b$. Let $\text{PATHS}_\geq(a, b)$ be the set of paths in $\text{PATHS}(a, b)$ which do not pass through points with $x > y$. How large is $\text{PATHS}_\geq(a, b)$?
1. \( \text{STRINGS}(a, b) = \{ x \in \{ R, U \}^* : x \text{ has } a \text{ R's and } b \text{ U's} \} \). \(^1\)

There is a natural bijection between \( \text{PATHS}(a, b) \) and \( \text{STRINGS}(a, b) \):

Path moves to Right, add \( R \) to sequence.  
Path goes up, add \( U \) to sequence.

So

\[
|\text{PATHS}(a, b)| = |\text{STRINGS}(a, b)| = \binom{a+b}{a}
\]

since to define a string we have state which of the \( a+b \) places contains an \( R \).

---

\(^1\{R, U\}^* = \text{set of strings of } R\text{'s and } U\text{'s} \)
2. Every path in $\text{PATHS}_>(a, b)$ goes through $(0,1)$. So

$$|\text{PATHS}_>(a, b)| = |\text{PATHS}((0, 1) \rightarrow (a, b))| - |\text{PATHS}_!((0, 1) \rightarrow (a, b))|.$$ 

Now

$$|\text{PATHS}((0, 1) \rightarrow (a, b))| = \binom{a+b-1}{a}$$

and

$$|\text{PATHS}_!((0, 1) \rightarrow (a, b))| = |\text{PATHS}((1, 0) \rightarrow (a, b))| = \binom{a+b-1}{a-1}.$$ 

We explain the first equality momentarily. Thus

$$|\text{PATHS}_>(a, b)| = \binom{a+b-1}{a} - \binom{a+b-1}{a-1}$$

$$= \frac{b-a}{a+b} \binom{a+b}{a}.$$
Suppose $P \in \text{PATHS}_{\neq}((0, 1) \rightarrow (a, b))$. We define $P' \in \text{PATHS}((1, 0) \rightarrow (a, b))$ in such a way that $P \rightarrow P'$ is a bijection.

Let $(c, c)$ be the first point of $P$, which lies on the line $L = \{x = y\}$ and let $S$ denote the initial segment of $P$ going from $(0, 1)$ to $(c, c)$.

$P'$ is obtained from $P$ by deleting $S$ and replacing it by its reflection $S'$ in $L$.

To show that this defines a bijection, observe that if $P' \in \text{PATHS}((1, 0) \rightarrow (a, b))$ then a similarly defined reverse reflection yields a $P \in \text{PATHS}_{\neq}((0, 1) \rightarrow (a, b))$. 
3. Suppose $P \in \text{PATHS}_{\geq}(a, b)$. We define $P'' \in \text{PATHS}_{>(a, b+1)}$ in such a way that $P \rightarrow P''$ is a bijection.

Thus

$$|\text{PATHS}_{\geq}(a, b)| = \frac{b - a + 1}{a + b + 1} \binom{a + b + 1}{a}.$$ 

In particular

$$|\text{PATHS}_{\geq}(a, a)| = \frac{1}{2a + 1} \binom{2a + 1}{a} = \frac{1}{a + 1} \binom{2a}{a}.$$ 

The final expression is called a Catalan Number.
The bijection

Given $P$ we obtain $P''$ by raising it vertically one position and then adding the segment $(0, 0) \rightarrow (0, 1)$.

More precisely, if $P = (0, 0), (x_1, y_1), (x_2, y_2), \ldots, (a, b)$ then $P'' = (0, 0), (0, 1), (x_1, y_1 + 1), \ldots, (a, b + 1)$.

This is clearly a $1-1$ onto function between $\text{PATHS}_{\geq}(a, b)$ and $\text{PATHS}_{>}(a, b + 1)$. 
Multi-sets

Suppose we allow elements to appear several times in a set: 
\{a, a, a, b, b, c, c, c, d, d\}.
To avoid confusion with the standard definition of a set we write 
\{3 \times a, 2 \times b, 3 \times c, 2 \times d\}.
How many distinct permutations are there of the multiset 
\{a_1 \times 1, a_2 \times 2, \ldots, a_n \times n\}? 
Ex. \{2 \times a, 3 \times b\}.

aabbb; ababb; abbab; abbaa; baabb
babab; babba; bbaab; bbaba; bbbaa.
Start with \( \{a_1, a_2, b_1, b_2, b_3\} \) which has \( 5! = 120 \) permutations:
\[ \ldots a_2 b_3 a_1 b_2 b_1 \ldots a_1 b_2 a_2 b_1 b_3 \ldots \]
After erasing the subscripts each possible sequence e.g. \( ababb \) occurs \( 2! \times 3! \) times and so the number of permutations is \( 5!/2!3! = 10 \).
In general if \( m = a_1 + a_2 + \cdots + a_n \) then the number of permutations is
\[
\frac{m!}{a_1!a_2!\cdots a_n!}
\]
Multinomial Coefficients

\[
\binom{m}{a_1, a_2, \ldots, a_n} = \frac{m!}{a_1! a_2! \cdots a_n!}
\]

\[
(x_1 + x_2 + \cdots + x_n)^m = \sum_{\substack{a_1 + a_2 + \cdots + a_n = m \\ a_1 \geq 0, \ldots, a_n \geq 0}} \binom{m}{a_1, a_2, \ldots, a_n} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.
\]

E.g.

\[
(x_1 + x_2 + x_3)^4 = \binom{4}{4, 0, 0} x_1^4 + \binom{4}{3, 1, 0} x_1^3 x_2 + \binom{4}{3, 0, 1} x_1^3 x_3 + \binom{4}{2, 1, 1} x_1^2 x_2 x_3 + \cdots
\]

\[
= x_1^4 + 4x_1^3 x_2 + 4x_1^3 x_3 + 12x_1^2 x_2 x_3 + \cdots
\]
Contribution of 1 to the coefficient of
\(x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}\) from every permutation in
\(S = \{x_1 \times a_1, x_2 \times a_2, \ldots, x_n \times a_n\}\).
E.g.
\[(x_1 + x_2 + x_3)^6 = \cdots + x_2 x_3 x_2 x_1 x_1 x_3 + \cdots\]
where the displayed term comes by choosing \(x_2\) from first
bracket, \(x_3\) from second bracket etc.

Given a permutation \(i_1 i_2 \cdots i_m\) of \(S\) e.g. \(331422 \cdots\) we choose
\(x_3\) from the first 2 brackets, \(x_1\) from the 3rd bracket etc.
Conversely, given a choice from each bracket which contributes
to the coefficient of \(x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}\) we get a permutation of \(S\).
Balls in boxes

$m$ distinguishable balls are placed in $n$ distinguishable boxes. Box $i$ gets $b_i$ balls.

$\# \text{ ways is } \binom{m}{b_1, b_2, \ldots, b_n}$.

$m = 7$, $n = 3$, $b_1 = 2$, $b_2 = 2$, $b_3 = 3$

No. of ways is

$$7!/(2!2!3!) = 210$$

$$\begin{bmatrix} 1, 2 \end{bmatrix} \begin{bmatrix} 3, 4 \end{bmatrix} \begin{bmatrix} 5, 6, 7 \end{bmatrix} \begin{bmatrix} 1, 2 \end{bmatrix} \begin{bmatrix} 3, 5 \end{bmatrix} \begin{bmatrix} 4, 6, 7 \end{bmatrix} \cdots \begin{bmatrix} 6, 7 \end{bmatrix} \begin{bmatrix} 4, 5 \end{bmatrix} \begin{bmatrix} 1, 2, 3 \end{bmatrix}$$

3 1 3 2 1 3 2
Ball 1 goes in box 3, Ball 2 goes in box 1, etc.
Conversely, given an allocation of balls to boxes:

\[
\begin{array}{cc}
3 & 7 \\
2 & 4 \\
1 & 5 \\
\end{array}
\]

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How many trees? – Cayley’s Formula

n=4

4 12

n=5

5 60 60

n=6

6 120 360 90

Basic Counting
Prüfer’s Correspondence

There is a 1-1 correspondence \( \phi_V \) between spanning trees of \( K_V \) (the complete graph with vertex set \( V \)) and sequences \( V^{n-2} \). Thus for \( n \geq 2 \)

\[
\tau(K_n) = n^{n-2}
\]

Cayley’s Formula.

Assume some arbitrary ordering \( V = \{v_1 < v_2 < \cdots < v_n\} \).

\( \phi_V(T) \):
begin
T_1 := T;
for i = 1 to n − 2 do
begin
s_i := neighbour of least leaf \( \ell_i \) of \( T_i \).
T_{i+1} = T_i - \ell_i.
end
\( \phi_V(T) = s_1s_2\ldots s_{n-2} \)
end
Basic Counting

6, 4, 5, 14, 2, 6, 11, 14, 8, 5, 11, 4, 2
Lemma

\( v \in V(T) \) appears exactly \( d_T(v) - 1 \) times in \( \phi_V(T) \).

Proof

Assume \( n = |V(T)| \geq 2 \). By induction on \( n \).

\( n = 2 \): \( \phi_V(T) = \Lambda = \text{empty string} \).

Assume \( n \geq 3 \):

\[ \phi_V(T) = s_1 \phi_{V_1}(T_1) \] where \( V_1 = V - \{s_1\} \).

\( s_1 \) appears \( d_{T_1}(s_1) - 1 + 1 = d_T(s_1) - 1 \) times – induction.

\( v \neq s_1 \) appears \( d_{T_1}(v) - 1 = d_T(v) - 1 \) times – induction.

□
Construction of $\phi^{-1}_V$

Inductively assume that for all $|X| < n$ there is an inverse function $\phi^{-1}_X$. (True for $n = 2$).

Now define $\phi^{-1}_V$ by

$$\phi^{-1}_V(s_1s_2\ldots s_{n-2}) = \phi^{-1}_{V_1}(s_2\ldots s_{n-2}) \text{ plus edge } s_1\ell_1,$$

where $\ell_1 = \min\{s \in V : s \notin \{s_1, s_2, \ldots, s_{n-2}\}\}$ and $V_1 = V - \{\ell_1\}$. Then

$$\phi_V(\phi^{-1}_V(s_1s_2\ldots s_{n-2})) = s_1\phi_{V_1}(\phi^{-1}_{V_1}(s_2\ldots s_{n-2}))$$

$$= s_1s_2\ldots s_{n-2}.$$

Thus $\phi_V$ has an inverse and the correspondence is established.
$n = 10$
$s = 5, 3, 7, 4, 4, 3, 2, 6.$
Number of trees with a given degree sequence

Corollary

If $d_1 + d_2 + \cdots + d_n = 2n - 2$ then the number of spanning trees of $K_n$ with degree sequence $d_1, d_2, \ldots, d_n$ is

$$\binom{n - 2}{d_1 - 1, d_2 - 1, \ldots, d_n - 1} = \frac{(n - 2)!}{(d_1 - 1)! (d_2 - 1)! \cdots (d_n - 1)!}.$$

Proof  From Prüfer’s correspondence this is the number of sequences of length $n - 2$ in which 1 appears $d_1 - 1$ times, 2 appears $d_2 - 1$ times and so on. □
Inclusion-Exclusion

2 sets:

\[ |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \]

So if \( A_1, A_2 \subseteq A \) and \( \overline{A_i} = A \setminus A_i, \ i = 1, 2 \) then

\[ |\overline{A}_1 \cap \overline{A}_2| = |A| - |A_1| - |A_2| + |A_1 \cap A_2| \]

3 sets:

\[ |\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| = |A| - |A_1| - |A_2| - |A_3| \\
+ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| \\
- |A_1 \cap A_2 \cap A_3|. \]
General Case

$A_1, A_2, \ldots, A_N \subseteq A$ and each $x \in A$ has a weight $w_x$. (In our examples $w_x = 1$ for all $x$ and so $w(X) = |X|$.)

For $S \subseteq [N]$, $A_S = \bigcap_{i \in S} A_i$ and $w(S) = \sum_{x \in S} w_x$.

E.g. $A\{4,7,18\} = A_4 \cap A_7 \cap A_{18}$.

$A_{\emptyset} = A$.

Inclusion-Exclusion Formula:

$$w \left( \bigcap_{i=1}^{N} \overline{A_i} \right) = \sum_{S \subseteq [N]} (-1)^{|S|} w(A_S).$$
Simple example. How many integers in $[1000]$ are not divisible by 5, 6 or 8 i.e. what is the size of $\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3$ below? Here we take $w_x = 1$ for all $x$.

$A = A_{\emptyset} = \{1, 2, 3, \ldots, \}$ \hspace{1cm} |A| = 1000
$A_1 = \{5, 10, 15, \ldots, \}$ \hspace{1cm} |A_1| = 200
$A_2 = \{6, 12, 18, \ldots, \}$ \hspace{1cm} |A_2| = 166
$A_3 = \{8, 16, 24, \ldots, \}$ \hspace{1cm} |A_3| = 125
$A_{\{1,2\}} = \{30, 60, 90, \ldots, \}$ \hspace{1cm} |A_{\{1,2\}}| = 33
$A_{\{1,3\}} = \{40, 80, 120, \ldots, \}$ \hspace{1cm} |A_{\{1,3\}}| = 25
$A_{\{2,3\}} = \{24, 48, 72, \ldots, \}$ \hspace{1cm} |A_{\{2,3\}}| = 41
$A_{\{1,2,3\}} = \{120, 240, 360, \ldots, \}$ \hspace{1cm} |A_{\{1,2,3\}}| = 8

$$|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| = 1000 - (200 + 166 + 125)$$
$$+ (33 + 25 + 41) - 8$$
$$= 600.$$
Derangements

A derangement of $[n]$ is a permutation $\pi$ such that

$$\pi(i) \neq i : i = 1, 2, \ldots, n.$$ 

We must express the set of derangements $D_n$ of $[n]$ as the intersection of the complements of sets.

We let $A_i = \{\text{permutations } \pi : \pi(i) = i\}$ and then

$$|D_n| = \left| \bigcap_{i=1}^n \overline{A_i} \right|.$$
We must now compute $|A_S|$ for $S \subseteq [n]$.

$|A_1| = (n - 1)!$: after fixing $\pi(1) = 1$ there are $(n - 1)!$ ways of permuting $2, 3, \ldots, n$.

$|A_{\{1,2\}}| = (n - 2)!$: after fixing $\pi(1) = 1, \pi(2) = 2$ there are $(n - 2)!$ ways of permuting $3, 4, \ldots, n$.

In general

$|A_S| = (n - |S|)!$
\[ |D_n| = \sum_{S \subseteq [n]} (-1)^{|S|} (n - |S|)! \]
\[ = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n - k)! \]
\[ = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \]
\[ = n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}. \]

When \( n \) is large,
\[ \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \approx e^{-1}. \]
Proof of inclusion-exclusion formula

\[ \theta_{x,i} = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases} \]

\[ (1 - \theta_{x,1})(1 - \theta_{x,2}) \cdots (1 - \theta_{x,N}) = \begin{cases} 1 & x \in \bigcap_{i=1}^{N} \overline{A_i} \\ 0 & \text{otherwise} \end{cases} \]

So

\[ w\left(\bigcap_{i=1}^{N} \overline{A_i}\right) = \sum_{x \in A} w_x (1 - \theta_{x,1})(1 - \theta_{x,2}) \cdots (1 - \theta_{x,N}) \]

\[ = \sum_{x \in A} \sum_{S \subseteq [N]} (-1)^{|S|} \prod_{i \in S} \theta_{x,i} \]

\[ = \sum_{S \subseteq [N]} (-1)^{|S|} \sum_{x \in A} w_x \prod_{i \in S} \theta_{x,i} \]

\[ = \sum_{S \subseteq [N]} (-1)^{|S|} w(A_S). \]
Euler’s Function $\phi(n)$.

Let $\phi(n)$ be the number of positive integers $x \leq n$ which are mutually prime to $n$ i.e. have no common factors with $n$, other than 1.

$\phi(12) = 4$.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$ be the prime factorisation of $n$.

$$\phi(n) = \left| \bigcap_{i=1}^{k} \overline{A_i} \right|$$

$A_i = \{x \in [n] : p_i \text{ divides } x\}$, \hspace{0.5cm} 1 \leq i \leq k.$
\[ |A_S| = \frac{n}{\prod_{i \in S} p_i} \quad S \subseteq [k]. \]

\[
\phi(n) = \sum_{S \subseteq [k]} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i}
\]

\[
= n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)
\]
Surjections

Fix $n, m$. Let
\[ A = \{ f : [n] \rightarrow [m] \} \]
Thus $|A| = m^n$. Let
\[ F(n, m) = \{ f \in A : f \text{ is onto } [m] \}. \]

How big is $F(n, m)$?
Let
\[ A_i = \{ f \in F : f(x) \neq i, \forall x \in [n] \}. \]
Then
\[ F(n, m) = \bigcap_{i=1}^{m} \overline{A_i}. \]
For $S \subseteq [m]$ 

$$A_S = \{ f \in A : f(x) \notin S, \forall x \in [n] \}.$$ 

$$= \{ f : [n] \rightarrow [m] \setminus S \}.$$ 

So 

$$|A_S| = (m - |S|)^n.$$ 

Hence 

$$F(n, m) = \sum_{S \subseteq [m]} (-1)^{|S|} (m - |S|)^n$$ 

$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m - k)^n.$$ 

Basic Counting
Scrambled Allocations

We have \( n \) boxes \( B_1, B_2, \ldots, B_n \) and \( 2n \) distinguishable balls \( b_1, b_2, \ldots, b_{2n} \).

An allocation of balls to boxes, two balls to a box, is said to be \textit{scrambled} if there does \textbf{not} exist \( i \) such that box \( B_i \) contains balls \( b_{2i-1}, b_{2i} \). Let \( \sigma_n \) be the number of scrambled allocations.

Let \( A_i \) be the set of allocations in which box \( B_i \) contains \( b_{2i-1}, b_{2i} \). We show that

\[
|A_S| = \frac{(2(n - |S|))!}{2^{n-|S|}}.
\]

Inclusion-Exclusion then gives

\[
\sigma_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2(n-k))!}{2^{n-k}}.
\]
First consider $A_\emptyset$:

Each permutation $\pi$ of $[2n]$ yields an allocation of balls, placing $b_{\pi(2i-1)}, b_{\pi(2i)}$ into box $B_i$, for $i = 1, 2, \ldots, n$. The order of balls in the boxes is immaterial and so each allocation comes from exactly $2^n$ distinct permutations, giving

$$|A_\emptyset| = \frac{(2n)!}{2^n}.$$ 

To get the formula for $|A_S|$ observe that the contents of $2|S|$ boxes are fixed and so we are in essence dealing with $n - |S|$ boxes and $2(n - |S|)$ balls.
Probléme des Ménages

In how many ways $M_n$ can $n$ male-female couples be seated around a table, alternating male-female, so that no person is seated next to their partner?

Let $A_i$ be the set of seatings in which couple $i$ sit together.

If $|S| = k$ then

$$|A_S| = 2k!(n - k)!^2 \times d_k.$$

$d_k$ is the number of ways of placing $k$ 1’s on a cycle of length $2n$ so that no two 1’s are adjacent. (We place a person at each 1 and his/her partner on the succeeding 0).

2 choices for which seats are occupied by the men or women. $k!$ ways of assigning the couples to the positions; $(n - k)!^2$ ways of assigning the rest of the people.
\[ d_k = \frac{2n}{k} \binom{2n - k - 1}{k - 1} = \frac{2n}{2n - k} \binom{2n - k}{k}. \]

(See slides 11 and 12).

\[
M_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \times 2k!(n-k)!^2 \times \frac{2n}{2n-k} \binom{2n-k}{k}
\]

\[= 2n! \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!. \]
The weight of elements in exactly $k$ sets:

Observe that

$$\prod_{i \in S} \theta_{x,i} \prod_{i \notin S} (1 - \theta_{x,i}) = 1 \text{ iff } x \in A_i, i \in S \text{ and } x \notin A_i, i \notin S.$$  

$W_k$ is the total weight of elements in exactly $k$ of the $A_i$:

$$W_k = \sum_{x \in A} \sum_{|S|=k} \sum_{i \in S} \theta_{x,i} \prod_{i \notin S} (1 - \theta_{x,i})$$

$$= \sum_{|S|=k} \sum_{x \in A} \sum_{i \in S} \theta_{x,i} \prod_{i \notin S} (1 - \theta_{x,i})$$

$$= \sum_{|S|=k} \sum_{T \supseteq S} \sum_{x \in A} \theta_{x,i} \prod_{i \in T} (1 - \theta_{x,i})$$

$$= \sum_{|S|=k} (-1)^{|T \setminus S|} w(A_T)$$

$$= \sum_{\ell=k}^{N} \sum_{|T|=\ell} (-1)^{\ell-k} \binom{\ell}{k} w(A_T).$$
As an example. Let $D_{n,k}$ denote the number of permutations $\pi$ of $[n]$ for which there are exactly $k$ indices $i$ for which $\pi(i) = i$. Then

$$D_{n,k} = \sum_{\ell=k}^{n} \binom{n}{\ell} (-1)^{\ell-k} \binom{\ell}{k} (n - \ell)!$$

$$= \sum_{\ell=k}^{n} \frac{n!}{\ell!(n-\ell)!} (-1)^{\ell-k} \frac{\ell!}{k!(\ell-k)!} (n - \ell)!$$

$$= \frac{n!}{k!} \sum_{\ell=k}^{n} \frac{(-1)^{\ell-k}}{(\ell-k)!}$$

$$= \frac{n!}{k!} \sum_{r=0}^{n-k} \frac{(-1)^{r}}{r!}$$

$$\approx \frac{n!}{ek!}$$

when $n$ is large and $k$ is constant.
Bonferroni Inequalities

For \( x \in \{0, 1, \ast\}^N \) let

\[ A_x = A_1^{(x_1)} \cap A_2^{(x_2)} \cap \cdots \cap A_n^{(x_N)}. \]

Here

\[ A_i^{(x)} = \begin{cases} A_i & x = 1 \\ \bar{A}_i & x = 0 \\ A & x = \ast \end{cases}. \]

So,

\[ A_{0,1,0,\ast} = \bar{A}_1 \cap A_2 \cap \bar{A}_3 \cap A = \bar{A}_1 \cap A_2 \cap \bar{A}_3. \]
Suppose that $X \subseteq \{0, 1, *\}^N$ and

$$\Delta = \Delta(A_1, A_2, \ldots, A_N) = \sum_{x \in X} \alpha_x |A_x|.$$  

Here $\alpha_x \in R$ for $\alpha_x \in X$.

**Theorem (Rényi)**

$\Delta \geq 0$ for all $A_1, A_2, \ldots, A_N \subseteq A$ iff $\Delta \geq 0$ whenever $A_i = A$ or $A_i = \emptyset$ for $i = 1, 2, \ldots, N$.

**Corollary**

$$\left| \bigcap_{i=1}^N \bar{A}_i \right| - \sum_{i=0}^k \sum_{|S| = i} (-1)^i |A_S| \begin{cases} \leq 0 & k \text{ even} \\ \geq 0 & k \text{ odd} \end{cases}$$
Proof of corollary: Suppose that \( A_1 = A_2 = \cdots = A_\ell = A \) and \( A_{\ell+1} = \cdots = A_N = \emptyset \). If \( \ell = 0 \) then \( \Delta = 0 \) and if \( 0 < \ell \leq N \) then

\[
\Delta = 0 - \sum_{i=0}^{k} (-1)^i \binom{\ell}{i} |A| \\
= |A| \begin{cases} 
0 & k \geq \ell \\
(-1)^{k+1} \binom{\ell-1}{k} & k < \ell.
\end{cases}
\]

where the identity

\[
\sum_{i=0}^{k} (-1)^i \binom{\ell}{i} = (-1)^k \binom{\ell - 1}{k}
\]

can be proved by induction on \( k \) for \( \ell \geq 1 \) fixed.
It follows from the corollary that if $D_n$ denotes the number of derangements of $[n]$ then

$$n! \sum_{i=0}^{2k-1} (-1)^i \frac{1}{i!} \leq D_n \leq n! \sum_{i=0}^{2k} (-1)^i \frac{1}{i!},$$

for all $k \geq 0$. 
Proof of Rényi’s Theorem: We begin by reducing to the case where $X \subseteq \{0, 1\}^N$. I.e. we get rid of *-components.

Consider $x = (0, 1, *, 1)$. We have

$$A_x = A_{(0,1,0,1)} \cup A_{(0,1,1,1)} \text{ and } A_{(0,1,0,1)} \cap A_{(0,1,1,1)} = \emptyset.$$ 

So,

$$|A_{(0,1,*,1)}| = |A_{(0,1,0,1)}| + |A_{(0,1,1,1)}|.$$ 

A similar argument gives

$$|A_{(*,1,*,1)}| = |A_{(0,1,0,1)}| + |A_{(0,1,1,1)}| + |A_{(1,1,0,1)}| + |A_{(1,1,1,1)}|.$$ 

Repeating this we can write

$$\Delta = \sum_{y \in Y} \alpha_y |A_y| \text{ where } Y \subseteq \{0, 1\}^N.$$ 

Basic Counting
We claim now that $\Delta(A_1, A_2, \ldots, A_N) \geq 0$ for all $A_1, A_2, \ldots, A_N \subseteq A$ iff $\alpha_y \geq 0$ for all $y \in Y$.

Suppose then that $\exists y = (y_1, y_2, \ldots, y_N) \in Y$ such that $\alpha_y < 0$. Now let

$$A_i = \begin{cases} A & y_i = 1. \\ \emptyset & y_i = 0. \end{cases}$$

Then in this case

$$\Delta(A_1, A_2, \ldots, A_N) = \alpha_y |A| < 0, \text{ contradiction.}$$

For if $y' = (y'_1, y'_2, \ldots, y'_N)$ and $y'_i \neq y_i$ for some $i$ then $A(y'_i) = \emptyset$ and so $A_{y'} = \emptyset$ too.